COMPARISON OF NUMERICAL METHODS TO CLOSED FORM SOLUTION FOR WAVE EQUATION ANALYSIS OF PILING[1]

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Abstract

This paper documents both the development of a closed form solution for the one-dimensional wave equation as it is applied to piles and its comparison to numerical solutions of the same problem. Wave mechanics have been used extensively in piles for many years but the solution of the wave equation has been almost exclusively a numerical one. The closed form solution used involves the solution of the semi-infinite pile solution immediately after impact and a Fourier series solution for times thereafter. This solution is compared with numerical solutions of different kinds for a given test case. The comparison shows variations between the closed form solution and the numerical methods that, although not egregious, are also not consistent from case to case. A wider variety of cases is needed to come to more general conclusions about the variations in these methods.

Introduction

Wave mechanics have been employed for the analysis of piles during impact driving for the last forty years. Many algorithms have been developed to perform this analysis, starting with Smith (1960). Any numerical method, however, will have inherent characteristics which will degrade its modelling of the actual physical system. These limitations have been documented in the past (Van Weele and Kay, 1984; Warrington, 1997), but are not widely appreciated by many users of the programs. The purpose of this paper is to compare results from some of these methods to a new closed form solution of the wave equation for piles. This will both highlight the differences in the methods and make for a greater understanding of the limitations of numerical methods.

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Outline of Closed Form Solution

The closed form solution used is the solution as proposed by Warrington (1997). An outline of this solution method follows.

Basic Solution Method

1. Determine the force-time or displacement-time history of the hammer at the pile top, either using semi-infinite pile theory or actual field data.
2. Solve the wave equation for the semi-infinite pile case. This is the solution for $t < L/c$, i.e., before the wave front propagates to the pile toe.
3. Compute the displacement and velocity functions as a function of distance at $t = L/c$. These become the initial conditions for the remainder of the problem.
4. Using the boundary conditions, compute the eigenvalues and eigenfunctions for the Fourier series. The pile top is assumed to be a free end in this case.
5. Using the displacement and velocity functions at $t = L/c$, compute the Fourier coefficients. This Fourier series is the solution for $t \geq L/c$.

As stated, this procedure assumes the transition point to be fixed at $t = L/c$. However, if the impulse force of the hammer system ends before this time, it is most advantageous to make the turnover point at the time when the impulse force becomes zero, or tension begins to develop in the pile top.

Assumptions for the Solution

The following assumptions are made:

1. The solution must be reasonably simple; the solution must not require integration or other transformation once it is formulated.
2. The system is a linear system. No plasticity is taken into account in this system.
3. All properties between the boundaries are uniform. These include pile area and material, dampening, soil spring constant.
4. Extensibility considerations of the pile top and toe are not significant. The validity of this assumption is dependent upon how the pile top force is formulated.
5. The force of the hammer is substantially finished before $t = L/c$. This solution favours long piles relative to the hammer blow duration.

Solution of the Wave Equation

The system being modelled is shown in Figure 1. The hammer has a rigid mass with a perfectly elastic cushion and another rigid mass to represent the pile cap. In this case the pile shaft has no soil interaction and the toe has only elasticity without dampening of any kind. The system is completely linear. Although dampening can be included in the hammer cushion, pile shaft and pile toe, and distributed elasticity along the shaft, these elements were left out both for the simplicity of the closed form solution and to examine specific difficulties with the numerical methods, such as difficulties with numerical methods, inaccuracies due to discretisation, and instabilities caused by boundary conditions.

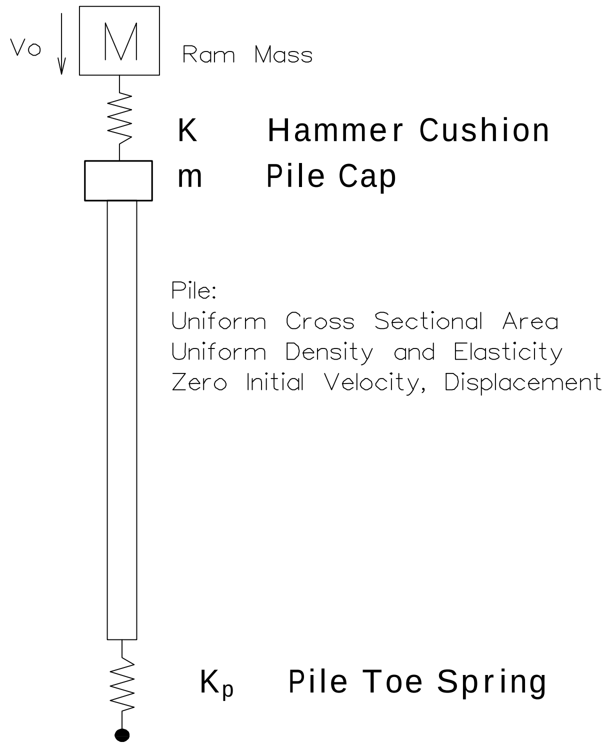


Figure 1 Hammer-Pile-Soil System

The basic equation for the system is

$$u_{tt}(x,t)=c^2u_{xx}(x,t) \quad (1)$$

where c = Acoustic Speed of Pile Material, m/sec
$u(x,t)$ = Displacement of Pile Particle, m
t = Time from Zero Point, seconds
x = Distance from Pile Top, m

The initial conditions are

$$u(x,0) = f(x) = 0 \quad (2)$$

and

$$u_t(x,0) = g(x) = 0 \quad (3)$$

where $f(x)$ = Initial Displacement Distribution in Pile, m
$g(x)$ = Initial Velocity Distribution in Pile, m/sec

The boundary condition for the pile toe is

$$-EAu_x(L,t) = K_t u(L,t) \quad (4)$$

where E = Young's Modulus for the pile, Pa
A = Pile cross-sectional area, m^2
K_t = Soil Toe Spring or Elastic Constant, N/m
L = Length of Pile, m

Before $t = L/c$, the boundary condition for the pile top is

$$u(0,t) = f(t) \quad (5)$$

where $f(t)$ = Displacement-time history of pile top, m

and after this time

$$u_x(0,t) = 0 \quad (6)$$

Solution of the Impact Hammer Force

To establish an expression for $f(t)$, we will use semi-infinite pile theory (Parola, 1970). The equations of motion for the ram and pile cap, respectively, are

$$Mx_r'' + K(x_r - x_t) = 0 \quad (7)$$

and

$$mx_t + Zx_t' + K(x_t - x_r) = 0 \quad (8)$$

where x_r = Displacement of ram, m
x_t = Pile Top Displacement, m
M = Ram Mass, kg
K = Hammer cushion stiffness, N/m
m = Pile Cap Mass, kg
Z = Pile Impedance, M-sec/m

Deeks and Randolph (1993) solved this system of equations. The resulting displacement-time history at the pile top can be applied to the rest of the semi-infinite pile by the equation (Kreyszig, 1993)

$$u(x,t) = f\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right) \quad (9)$$

where $H\left(t - \frac{x}{c}\right), H(t)$ = Heaviside Step Function

If we substitute the expressions

$$w_1 = \sqrt{\frac{K}{M}} \quad (10)$$

$$K_p = \frac{EA}{L} = \frac{Zc}{L} \quad (11)$$

$$w_2 = \frac{c}{L} \quad \text{(12)}$$

$$f(t) = x_t \quad \text{(13)}$$

where w_1 = Natural frequency of hammer-cushion system, rad/sec

w_2 = Natural frequency of pile, rad/sec

K_p = Pile static stiffness, N/m

and apply the solution to Equations (7) and (8) for the pile top to Equation (9), the displacement for the pile before t = L/c is

$$u(x,t) = \frac{V_0 M w_1^2}{m} \frac{\left(\left(e^{a_1\left(t-\frac{x}{Lw_2}\right)} - 1 \right)\left(a_3^2 + a_3 a_2^2\right) + e^{a_2\left(t-\frac{x}{Lw_2}\right)} \left(\begin{array}{l} \sin\left(a_3\left(t - \frac{x}{Lw_2}\right)\right)\left(a_1 a_2^2 - a_1^2 a_2 - a_1 a_3^2\right) \\ + \left(\cos\left(a_3\left(t - \frac{x}{Lw_2}\right)\right) - 1\right)\left(a_3 a_1^2 - 2a_1 a_2 a_3\right) \end{array} \right) \right)}{\left(a_1 a_3\left(a_2^2 + a_3^2\right)\left(a_1^2 - 2a_1 a_2 + a_2^2 + a_3^2\right)\right)} H\left(t - \frac{x}{Lw_2}\right) \quad \text{(14)}$$

and the velocity

$$u_t(x,t) = \frac{V_0 M w_1^2}{m} \frac{\left(a_1 e^{a_1\left(t-\frac{x}{Lw_2}\right)}\left(a_3^2 + a_3 a_2^2\right) + e^{a_2\left(t-\frac{x}{Lw_2}\right)} \left(\begin{array}{l} a_3 \cos\left(a_3\left(t - \frac{x}{Lw_2}\right)\right)\left(a_1 a_2^2 - a_1 a_3\right) \\ + \left(\sin\left(a_3\left(t - \frac{x}{Lw_2}\right)\right) - 1\right) \\ \left(a_1 a_2 a_3^2 + a_1 a_2^3 - a_1^2\left(a_2^2 + a_3^2\right)\right) \end{array} \right) \right)}{\left(a_1 a_3\left(a_2^2 + a_3^2\right)\left(a_1^2 - 2a_1 a_2 + a_2^2 + a_3^2\right)\right)} H\left(x - \frac{x}{Lw_2}\right) \quad \text{(15)}$$

where

$$a_1 = \frac{a_4 - 2K_p}{6mw_2} - \frac{2a_5}{3} \quad \text{(16)}$$

$$a_2 = -\frac{a_4 + 4K_p}{12 m w_2} + \frac{a_5}{3} \qquad (17)$$

$$a_3 = \frac{\sqrt{2}}{3}\left(\frac{a_4}{6 m w_2} + \frac{2 a_5}{3}\right) \qquad (18)$$

$$a_4 = \sqrt[3]{\begin{array}{l} 36 K_p m M w_1^2 w_2^2 - 72 K_p m^2 w_1^2 w_2^2 - 8 K_p^3 + \\ 12\sqrt{3} w_1 w_2 m \sqrt{\begin{array}{l} 4(w_1 w_2)^4 m M^3 + 12 w_1^2 w_2^2 m^2 M^2 - \\ (w_1 w_2)^2 M^2 K_p^2 + 12 (w_1 w_2)^4 m^3 M - \\ 20 (w_1 w_2)^2 m M K_p^2 + 4 (w_1 w_2 m)^4 + \\ 8 (w_1 w_2 m K_p)^2 + 4 K_p^4 \end{array}} \end{array}} \qquad (19)$$

$$a_5 = \frac{3(w_1 w_2)^2 m M + 3(w_1 w_2 m)^2 - K_p^3}{m w_2 a_4} \qquad (20)$$

If these equations are evaluated at t = L/c, they become the initial conditions f(x) and g(x) for the Fourier series. For the time t > L/c, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{l_n x}{L}\right)\left(C_1 \cos\left(l_n w_2\left(t - \frac{L}{c}\right)\right) + C_2 \sin\left(l_n w_2\left(t - \frac{L}{c}\right)\right)\right) \qquad (21)$$

and

$$u_t(x,t) = w_2 \sum_{n=1}^{\infty} l_n \cos\left(\frac{l_n x}{L}\right)\left(-C_1 \sin\left(l_n w_2\left(t - \frac{L}{c}\right)\right) + C_2 \cos\left(l_n w_2\left(t - \frac{L}{c}\right)\right)\right) \qquad (22)$$

The coefficients C_1 and C_2 are evaluated using the initial conditions given above according to the method of Tolstov (1962); an example of this for a similar case is given by Warrington (1997). The coefficients are omitted for brevity, as they are extremely involved. More important are the eigenvalues; they are the solution of the equation

$$\tan(l_n) = \frac{1}{l_n}\frac{K_s}{K_p}, p\left(n - \frac{3}{2}\right) < l_n < p\left(n - \frac{1}{2}\right), n = 1,2,3,\ldots\infty \qquad (23)$$

The values for l_n are transcendental and can only be solved by successive solutions of this equation for the given intervals. In many Fourier series, the eigenvalues are an integer multiple of a given value, which produces a regular set of harmonics from the fundamental frequency. In this case, the pile toe boundary condition generally does not allow for this.

This inharmonicity (Benade, 1976) is dependent upon the ratio of the stiffness of the pile toe soil to the overall stiffness of the pile. Since the pile stiffness is always real and positive, there are three cases to consider, depending upon the pile toe stiffness:

1. The pile toe stiffness is zero (free end.) In this case the tangent of l_n is always zero and Equation (23) reduces to

$$l_n = np, n = 1,2,3,\ldots\infty \qquad (24)$$

1. The pile toe stiffness is infinite (fixed end.) For this the tangent of l_n is infinite and Equation (23) reduces to

$$l_n = (2n-1)p, n = 1,2,3,\ldots\infty \qquad (25)$$

3. The pile toe stiffness is positive, real and finite; in this case Equation (23) must be solved for each of the periods.

This is an important result because it shows that the resonant frequency of a pile is not strictly according to Equation (10). It also shows that this solution of the wave equation can be applied to a wide variety of idealizations of the system.

Description of the Numerical Methods

Now that the closed form solution has been formulated, the numerical methods that it is to be compared with are described. These are described below. A more detailed description of these methods, including information on specifics of the application of these methods to this type of problem, are given in Warrington (1997).

Direct Stiffness Solution using Maple V Release 4

For most engineering problems such as this, when one considers the use of a numerical method, the first idea that comes to mind is the finite element method. For this paper the general purpose mathematical software package Maple V Release 4 was employed to construct and use a direct stiffness model of the pile for the undamped case. This model was constructed using Newmark's method as described by Logan (1992). The semi-infinite pile model was used to generate the force-time function at the pile top, as also for the closed form solution. Although Maple is not the most efficient code for this application, its matrix manipulation capabilities (it can do this symbolically in some cases) make this code relatively simple to use for the purpose.

Direct Stiffness Solution using ANSYS-ED 5.3 with Commercial License

One interesting concept that has not been widely pursued either by researchers or practitioners has been the use of general purpose finite element codes for stress wave analysis of piles. For both undamped and damped cases the closed form solution was compared with results from the ANSYS general purpose computer program. The pile top force can be simulated either by applying a force-time relationship or simulating the drop of a mass onto the hammer cushion. Both of these were modelled in ANSYS for this paper.

Finite Difference Solution using WEAP87

From both an historical and a practical standpoint, the most important comparison is with the finite difference techniques that have been the industry standard since the days of Smith (1960). For this purpose the WEAP87 program was used. This is similar to the WEAP86 program as described by Goble and Rausche (1986). This program has a relatively undemanding personal computer implementation and many options for input and output. These are necessary in this case as the entry of soil parameters that are similar to those used in the closed form solution require some care because their theoretical basis is different..

Example Case for Comparison Purposes

Statement of the Problem

The basic problem under consideration is the driving of a 1000 mm diameter steel pipe pile, 50 m long, with a wall thickness of 40 mm. The hammer used has a ram mass of 15 metric tons; it has an equivalent stroke of 1.5 m and a mechanical efficiency of 80%. The cushion block has a stiffness of 2.45 GN/m and has no damping. All of the soil resistance is represented as an elastic spring at the pile toe; its stiffness is the same as the pile's. This example was analysed for $0 < t < 4L/c$ for displacement-time and force-time histories at the pile top (L = 0 m), pile middle (L = 25 m) and pile toe (L = 50m).

Values for the variables of the solution are shown in Warrington (1997). These are either given variables or computed using the appropriate equations given earlier.

Computation of Pile Top Force and Displacement

The closed form solution is dependent upon the function of the pile top force, as is the Newmark method using Maple V and most of the ANSYS runs.

For the case considered, the displacement-time history for this is what is derived in the closed-form solution. Substituting the variables and solving for x=0, the result is

$$f(t) = .0149 - .01339e^{-835.176t} - .00151e^{-396.154t}\cos(401.6678t) - .0293e^{-396.154t}\sin(401.6678t) \qquad (26)$$

However, for both the Newmark's Solution using Maple V and the ANSYS solution using an applied force, it is more convenient to use a force-time history rather than a displacement-time history. This can be found by multiplying the result of Equation (15) by the pile impedance and substituting again x = 0. In this case the for-time history at the pile top is

$$F_0(t) = 54.61 \times 10^6 \left(e^{-835.176t} - e^{-396.154t}\cos(401.6678t)\right) + 59.69 \times 10^6 e^{-396.154t}\sin(401.6678t) \quad (27)$$

Two items need to be noted for these results:

- The semi-infinite pile top force-time curve is substantially complete at about t = L/c, so this criterion of the closed form solution is met.
- The negative portions are physically impossible because the hammer cushion is inextensible; however, these are not very significant.

Presentation and Discussion of the Results

Because of the nature of the results, graphical comparison is the most expedient method to view these results. The displacement-time and force-time histories are compared in two ways: a) between differing places on the pile for a single method, and b) between methods for given points on the pile.

General Comments

In comparing the results, the first thing that needs to be observed is that the solution methods can be grouped in several ways. These groupings are important to understand in order to properly interpret the result.

Type of Solution: The two-stage closed form solution is obviously the only solution of its type, as opposed to the other, numerical solutions. However, the numerical solutions themselves are different, as Newmark's Solution using Maple V and ANSYS are one the one hand finite element solutions and WEAP87 is a finite difference solution.

Pile Top Loading: The closed from solution uses Equation (26) for the pile top displacement. Newmark's Solution using Maple V, and ANSYS with an applied force use Equation (27) for the pile top force. WEAP87 and ANSYS with an impacting mass, however, actually model the motion the ram, hammer cushion and pile cap on the pile top. This enables these methods to model secondary impacts of the ram on the cushion material, which do not appear in the assumed loading methods. These methods also can model ram-cushion and pile cap-pile separation. In this case, the effects of omitting this are not significant, but they can be (Deeks and Randolph, 1993). This is an illustration of one of the advantages of the numerical solutions over the closed form ones.

Anti-Oscillatory Dampening: Many numerical methods include some of this to eliminate parasite oscillations due to discretisation (Bossard and Corté, 1983). ANSYS and WEAP87 both have this included. Newmark's Method using Maple V does not. The closed form solution does not need this type of dampening; however, it is important with a Fourier series to include sufficient terms to accurately model the system (sixty-five terms were used in this case.)

Displacements

Figure 2 shows the displacement-time histories by comparing the three pile locations using the same method for each graph, and Figure 3 shows these histories by comparing the methods with each other at each pile location.

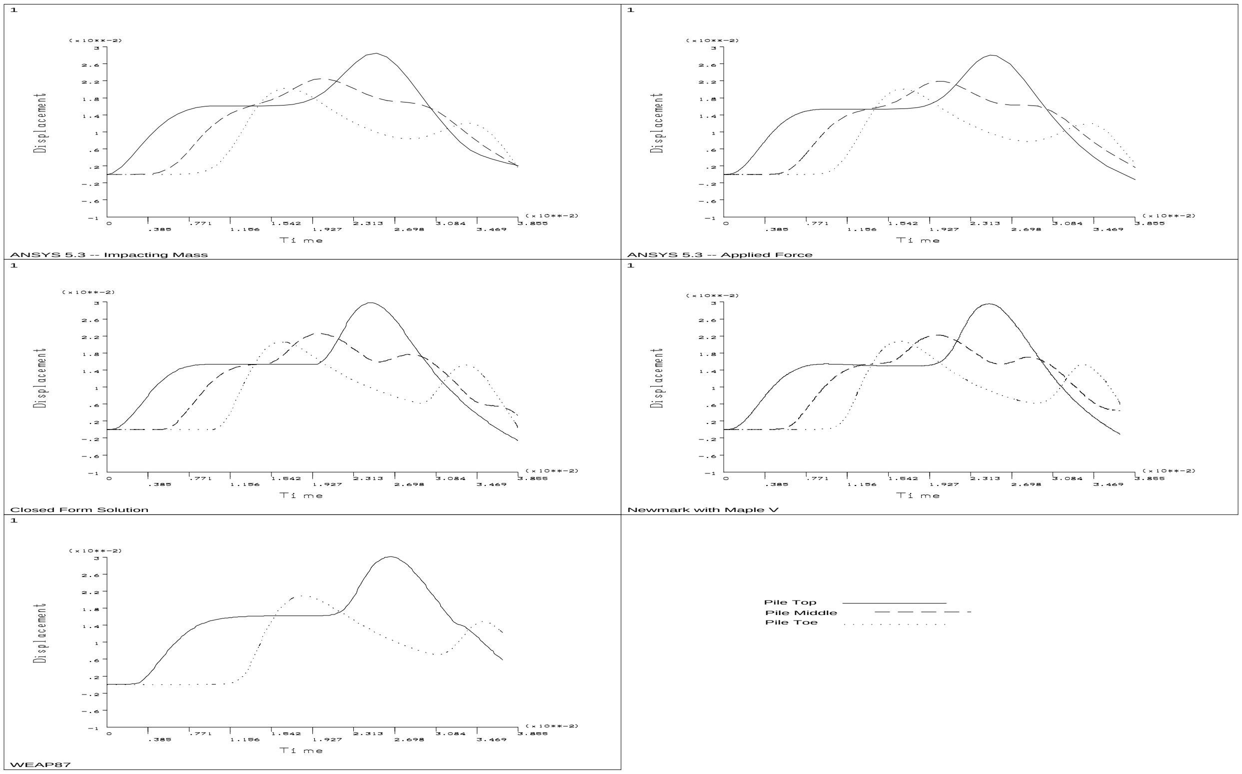


Figure 2 Comparison of Pile Locations, Displacements

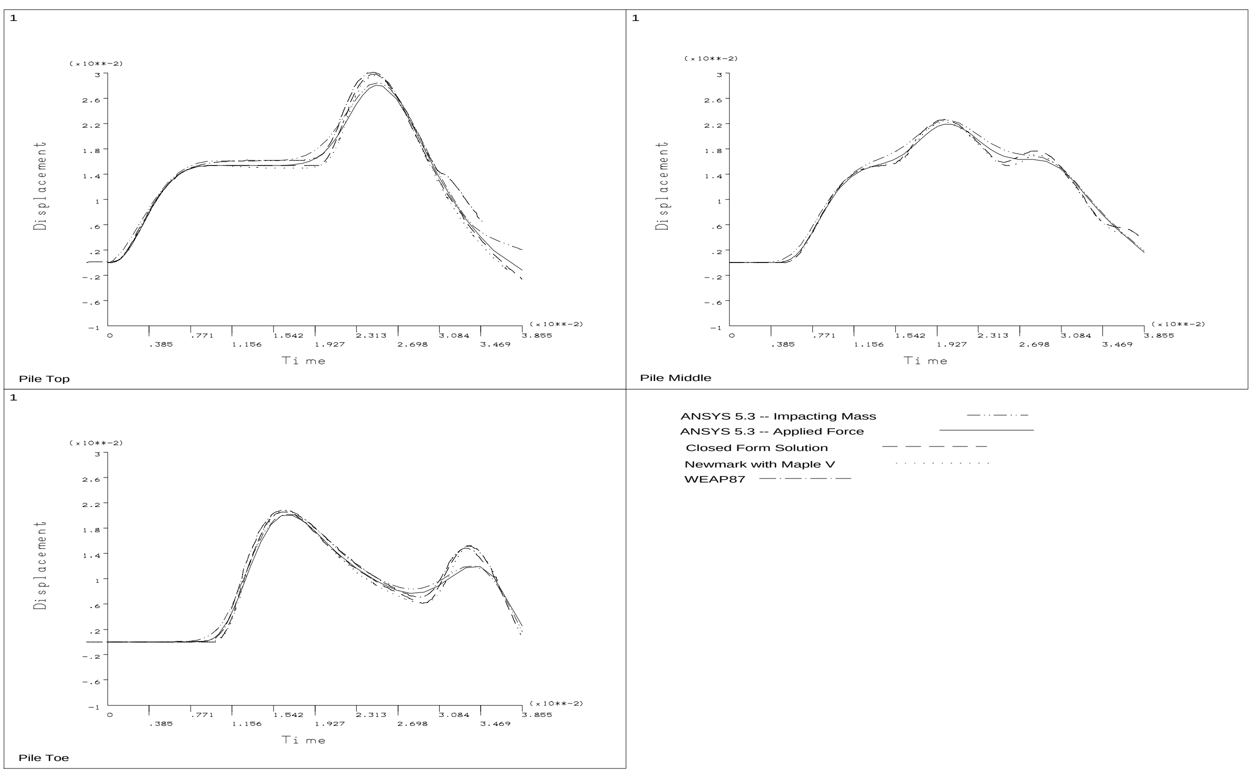


Figure 3 Comparison of Methods, Displacements

These plots first show the classic pattern of wave propagation: pile top movement begins at time t = 0, a similar displacement pattern begins for the pile middle at t = L/2c, and movement at the toe begins at t = L/c.. The following observations can be made in comparing the solutions:

- All of the solutions are in general agreement concerning the solution of the displacement; there are no solutions which significantly vary from the closed form solution.
- Both of the solutions that modelled the impacting mass of the ram (ANSYS w/Impacting Mass, WEAP87) have noticeable variations in the pile top displacement for $t \geq 3L/c$. This is due to a secondary impact and will be discussed in more detail when the forces are considered.
- Both of the ANSYS solutions consistently tend to “miss” the peaks in the displacement-time history, especially in the secondary rebound of the pile toe. The other numerical methods model this well.
- All of the numerical solutions show evidence of displacement in advance of the arrival of the stress wave. For example, at the pile toe there should be no displacement before the closed form solution records it, i.e. at t = L/c. However all of the numerical solutions show some evidence of this.

Forces

Figure 4 shows the force-time histories for the undamped case by comparing the three pile locations using the same method for each graph, and Figure 5 shows these histories by comparing the methods with each other at each pile location.

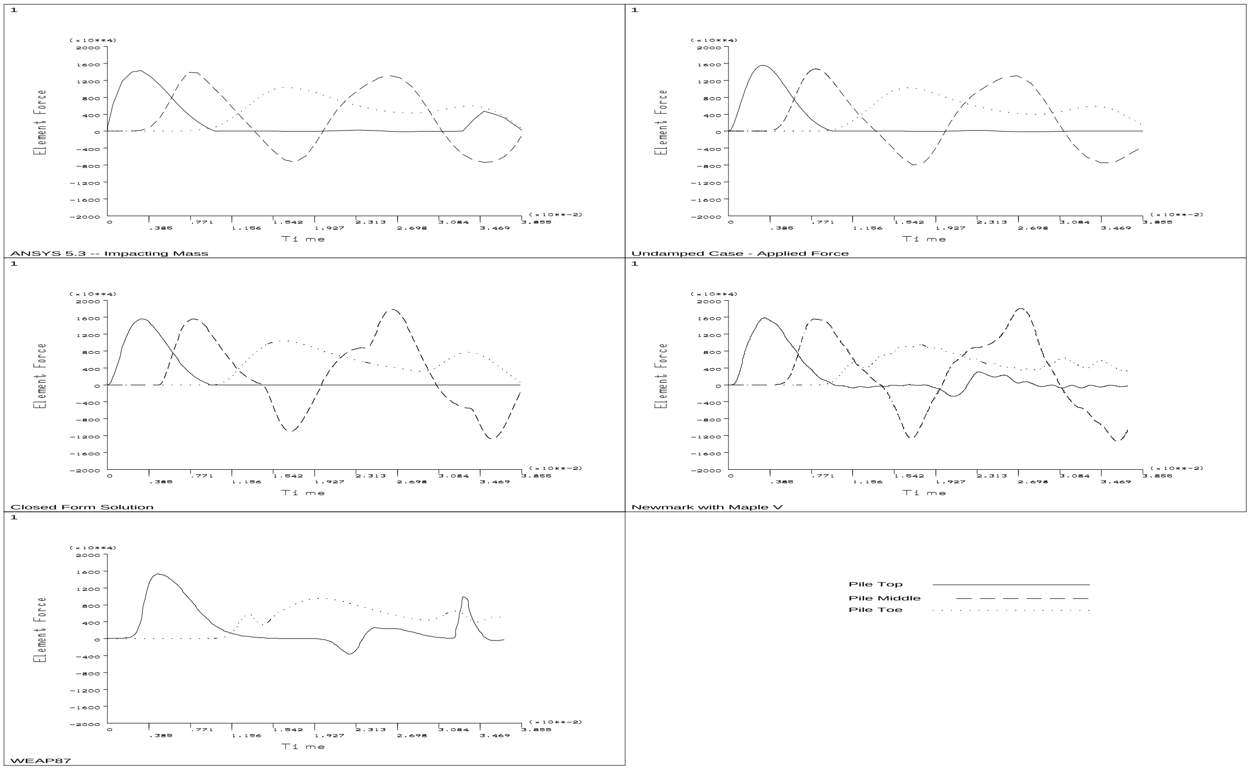


Figure 4 Comparison of Pile Locations, Forces

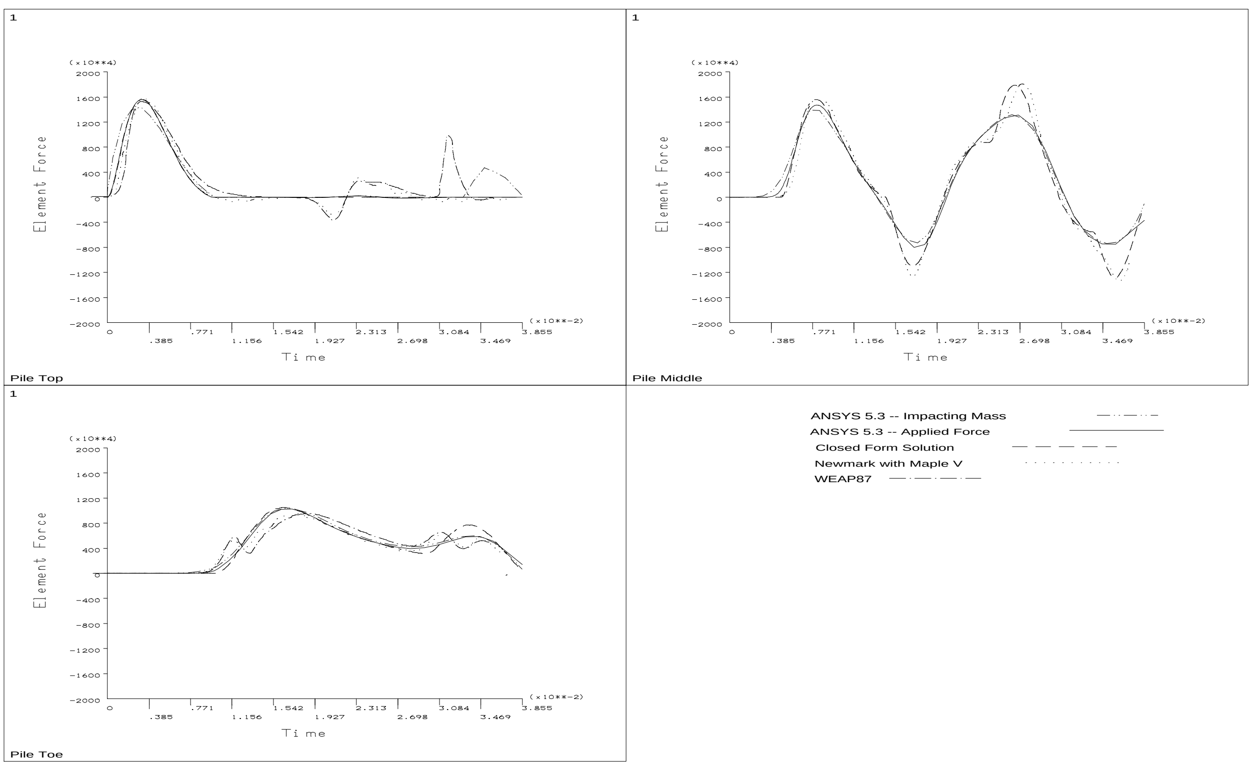


Figure 5 Comparison of Methods, Forces

The general wave propagation pattern is the same here as the displacements. The following observations can be made about these results:

- As with the displacements, the general nature of the results is reasonably consistent with the various methods, although the variations are more significant with the forces than with the displacements.
- The highest forces -- both tensile and compressive -- take place in the middle of the pile, where the effect of the boundary conditions is minimized.
- Only Newmark's Method with Maple V really replicate the high peak forces in the middle of the pile. ANSYS tends to "round" these forces off. WEAP87 did not report any results for the pile middle in spite of instructions to do so.
- Both ANSYS with an impacting mass and WEAP87 show a marked peak force at the pile top with a secondary impact; however, both the intensity and timing of the peak force is different.
- The pile toe showed considerable differences in the details of the force-time histories amongst the methods. Grouping methods with similar results comes up with some interesting combinations. The ANSYS solutions are very close, as one would expect; however, Newmark's Method with Maple V and WEAP87 are also close and both of these pairs vary from the closed form solution. This grouping replicates itself at around $t=2L/c$ at the pile top, as both of these methods experience the same spurious oscillation at this point.

Discussion

In most cases numerical solutions cannot be expected to render an "exact" solution to any physical problem; however, they are used frequently in problems because they are able to model systems that cannot be modelled completely in closed form. The complete understanding of the results of any numerical method, however, requires some appreciation of the errors inherent in the solution technique. These have been documented in some cases with numerical methods applied to this problem but many end users are unaware of their existence, and in any case cannot check the results against another solution technique.

The first thing to note about the comparison presented above is that it is incomplete. The nature of the closed form solution makes it impossible to replicate all of the parameters of the numerical solutions. These include a) nonlinearities, especially soil plasticity, b) energy dissipative elements (which can be modelled to some extent in closed form but not completely,) and c) separation factors at the boundaries. However, use of the undamped wave equation in closed form for comparison with numerical methods can allow the examination of certain difficulties in numerical methods, such as basic integration and discretisation induced errors and stability considerations. These are basic to the integrity of any numerical solution.

Having said this, none of the numerical methods examined in this case showed any "egregious" errors with respect to the closed form solutions. In this case the ANSYS runs were the least satisfactory in comparison to the closed form solution, although they compared well with each other, which showed that actually modelling the hammer induced few if any errors in the solution (as opposed to an applied force.) WEAP87 and Newmark's Method with Maple V both correlated well with each other and reasonably well with the closed form solution.

However, it must be noted that this is not the first time this type of comparison has been done; it was performed with similar (but not identical) cases in Warrington (1997). In this comparison with the undamped case Newmark's Method with Maple V showed parasite oscillations; with the damped case, WEAP87 experienced difficulties as well. With these comparisons ANSYS performed the best.

This leads to an important result, namely that variations between numerical and closed form solutions not only depend on the methods but also on the problem itself. Any meaningful use of the closed form solutions as a comparison tool needs to involve a large number of individual cases in addition to a large array of numerical integration algorithms.

Conclusion

The comparison of the three numerical methods (ANSYS, Newmark's Method with Maple V and WEAP87) show a reasonable comparison in the case examined to the closed form solution. There were variations between both the numerical methods and the closed form solution and among the numerical methods; however, the nature of the problem precludes generalities concerning the nature of the errors for a wide variety of cases. A better understanding of these variations requires a broader study with a larger number of cases.

References

BENADE, A.H. (1976) *Fundamentals of Musical Acoustics*. New York: Dover Publications, Inc..

BOSSARD, A., and CORTÉ, J.-F. (1983*) Battage des pieux -- programme de calcul Batlab*. Paris: Laboratoire Central des Ponts et Chauseés.

DEEKS, A.J., and RANDOLPH, M.F. (1993) "Analytical Modelling of Hammer Impact for Pile Driving." *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 17, pp. 279-302.

GOBLE, G.G., and RAUSCHE, F. (1986) "Wave Equation Analysis of Pile Foundations, WEAP86 Program." Federal Highway Administration Report Contract DTFH61-84-C-00100. Washington: Federal Highway Administration.

KREYSZIG, E. (1993) *Advanced Engineering Mathematics*. Seventh Edition. New York: John Wiley & Sons, Inc.

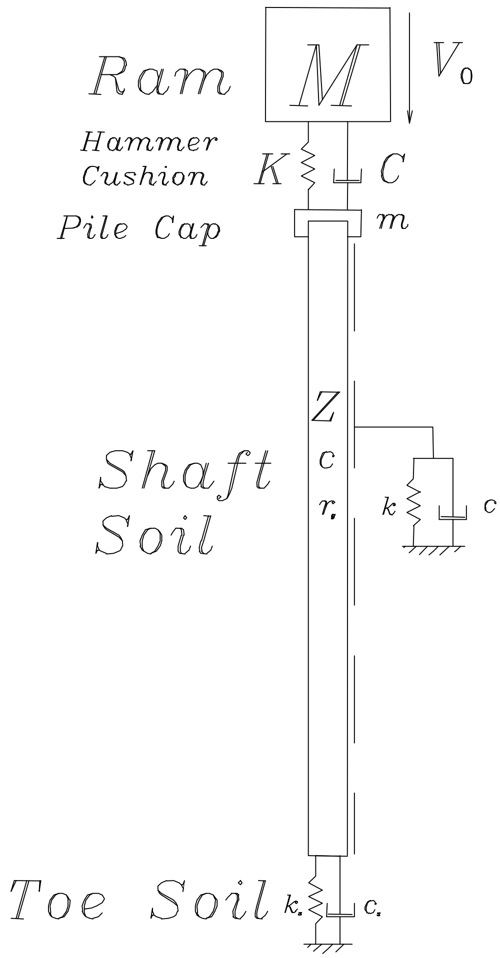
LOGAN, D.L. (1992) *A First Course in the Finite Element Method.* Boston: PWS Publishing Company.

SMITH, E.A.L. (1960) "Pile-Driving Analysis by the Wave Equation*." Journal of the Soil Mechanics and Foundations Division*, August, pp. 35-61. New York: American Society of Civil Engineers.

TOLSTOV, G.P. (1962) *Fourier Series*. Englewood Cliffs, NJ: Prentice-Hall, Inc..

VAN WEELE, A.F., and KAY, S. (1984) "Analytical Results with Numerical Programs." *Proceedings of the Second International Conference on the Application of Stress-Wave Theory On Piles.* Rotterdam: A.A. Balkema.

WARRINGTON, D.C. (1997) *Closed Form Solution of the Wave Equation for Piles*. Master's Thesis, University of Tennessee at Chattanooga.



Now that x=L…

For those of us involved in the one-dimensional wave equation, this means that you have reached the end! We trust that the information presented in the article concerning the wave equation or other technical matters has been useful to you. We should now like to take the time to make some other observations.

It is our conviction that the beauty of our world and universe, especially as it is expressed mathematically but certainly in other ways, speaks of its formation by an intelligent Creator. This is underscored by the unity that appears both in mathematics and in the physical laws which mathematics are used in to quantify and qualify. As scientists and engineers we depend upon this unity to both make sense out of what we observe and to make progress both in our knowledge and in our application of that knowledge to practical problems.

But as we turn away from the reverie of beautiful formulations, we see a world that is marred by human failing. This manifests itself in many forms that we are reminded of daily. The longer we live on this earth the more those failings come home to inflict pain upon us, no matter how hard we try to escape them.

It was not God's intent to leave us with this pain alone in his creation but to offer us a way by which we finite beings be united into his perfect infinity, something which is both definable and beyond definition. In infinity past he was with his Son Jesus Christ and Jesus came to live amongst us, share our situation and ultimately face torture and execution by those who were threatened by his message.

But this was not the end, for Jesus being God rose from the dead and offers us both a way out of our present condition in this life and eternal life with God, not by simply following a set of rules but by having God himself live in us and both empowering and leading us in a better way. If we commit ourselves to Jesus then for us $L = \infty$, which means that we have life forever.

All of these things are described in the book called the *Bible*; but in the meanwhile you can learn more at the website

http://www.geocities.com/penlay

or by emailing us at uttc2uxx@ geocities.com. We look forward to hearing from you.