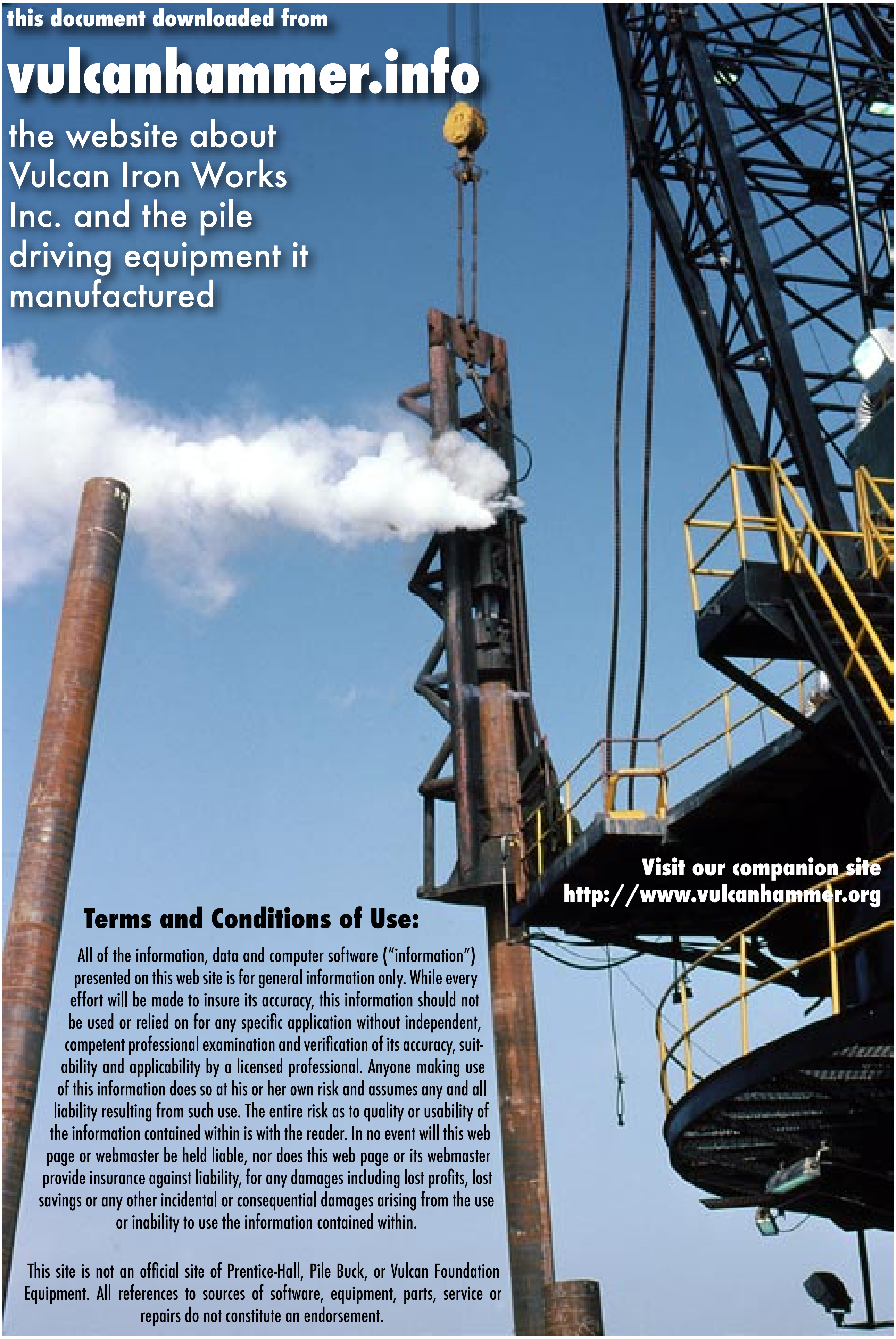

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COMPUTER STUDY OF DYNAMIC BEHAVIOR OF PILING[a]

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SYNOPSIS

The application of wave theory to the investigation of structural behavior of piling is examined. A digital-computer program based substantially on the work of E. A. L. Smith was used in generating the theoretical solutions. The essentials of Smith's development are given, followed by a discussion of certain extensions and applications of the procedure.

It is illustrated how, through the use of high-speed digital computation, the influence on pile behavior of factors such as ram weight, ram velocity, diesel-hammer pressure, capblock and cushion block stiffnesses, pile material properties, and soil properties may be evaluated.

Comparisons are made with the "exact" solution for an ideal bar and with experimental results from a field test. The effects of segment length and time interval for the discrete-element solution are examined. The use of the automatic plotting capability of the digital computer is illustrated.

Note.—Discussion open until January 1, 1964. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 89, No. ST4, August, 1963.

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INTRODUCTION

The study of behavior of piling has received considerable attention by investigators in the past. Much work has been directed toward establishing simplified formulas, both empirical and semi-rational. It is generally recognized that none of these proves completely satisfactory for the broad spectrum of pile types, pile drivers, and soil conditions encountered in present-day foundation problems. For a discussion of pile formulas, in general, the reader is referred to the work of Robert D. Chellis.[4]

The purpose of the present paper is to examine recent developments in the use of wave theory in the structural analysis of piling. In the work of Smith[5] a major contribution has been made toward adapting the general theory of stress-wave propagation to the peculiar problems associated with piling. One of the major objections that has been raised against this approach is the need for high-speed electronic computation in the practical application of the procedure. In the present paper, the writers endeavor to demonstrate that considerable useful data can be developed through generalized parameter studies. Certain extensions and refinements to Smith's work are also described. It is emphasized that the present study is limited to structural behavior of piling during driving. An effort has not been made to correlate resistance to penetration with ultimate bearing capacity.

D. V. Isaacs[6] is thought to be the first (1931) to note the occurrence of wave action in piling during driving. However, it is interesting to note that Isaac Todhunter and Karl Pearson[7] state that H. Moseley[8] in 1843 gives driving of piles as a special application of impact and resilience theory.

In 1938, W. H. Glanville, G. Grime, E. N. Fox, and W. W. Davies[9] published the results of extensive mathematical and experimental studies of piling in which stress-wave propagation was considered. Because of limited computational capability at that time, the application of wave theory necessarily involved simplifying assumptions. Nevertheless, this work has considerable value. In 1940 A. E. Cummings[10] examined dynamic pile-driving formulas, in general, and provided a brief description of the wave-theory approach and the theoretical work of Glanville and his associates.

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4 "Pile Foundations," by Robert D. Chellis, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1961.

5 "Pile-driving Analysis by the Wave Equation," by E. A. L. Smith, Transactions, ASCE, Vol. 127, 1962, Part I, p. 1145.

6 "Reinforced Concrete Pile Formulae," by D. V. Isaacs, Transactions, Institution of Engineers, Australia, Vol. 12, 1931, p. 312.

7 "A History of the Theory of Elasticity," by Isaac Todhunter and Karl Pearson, Dover Publications, Inc., New York, N. Y., Vol. 1, 1960, p. 669.

8 "The Mechanical Principles of Engineering and Architecture," by H. Moseley, London, 1843, pp. 598-603.

9 "An Investigation of the Stresses in Reinforced Concrete Piles during Driving," by W. H. Glanville, G. Grime, E. N. Fox, and W. W. Davies, British Building Research Board Technical Paper No. 20, Dept. of Scientific and Industrial Research, His Majesty's Stationery Office, London, 1938.

10 "Dynamic Pile Driving Formulas," by A. E. Cummings, Journal of the Boston Society of Civil Engineers, January, 1940.

Smith[5] has adapted wave theory in a more realistic manner to the actual conditions met in pile driving. He published considerable previous work[11,12,13] leading to the development of his procedure.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in the Appendix.

THEORY

Wave Equation.—The wave-theory approach to the problem does not involve a "formula" in the usual sense. The basis for the procedure is the classical one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

in which c represents the velocity of propagation of strain wave along bar $(=\sqrt{E/\rho})$; x is the direction of longitudinal axis; u represents the displacement of bar cross section in x direction; t denotes time; E is the modulus of elasticity of material; and ρ is the mass per unit volume of material.

Wave theory as applied to problems of longitudinal wave transmission and impact is investigated in detail by L. H. Donnell,[14] S. Timoshenko and J. N. Goodier,[15] H. Kolsky,[16] and H. N. Abramson, H. J. Plass, and E. A. Ripperger.[17] Julius Miklowitz[18] provides an extensive list of references on the broad subject of elastic wave propagation.

Smith's Idealization.—Fig. 1 illustrates the idealization of the pile system suggested by Smith. In general, the system is considered to be composed of [see Fig. 1(a)] the following: 1. A ram, to which an initial velocity is imparted by the pile driver; 2. a capblock (cushioning material); 3. a pile cap; 4. a cushion block (cushioning material); 5. a pile; and 6. the supporting medium, or soil. Fig. 1(b) shows the idealizations for the various components of the actual pile. The ram, capblock, pile cap, cushion block, and pile are pictured as appropriate discrete weights and springs. The frictional soil resistance on the side of the pile is represented by a series of side springs; the

11 "Pile Driving Impact," by E. A. L. Smith, Proceedings, Industrial Computation Seminar, September, 1950, International Business Machines Corp., New York, N. Y., 1951, p. 44.

12 "Impact and Longitudinal Wave Transmission," by E. A. L. Smith, Transactions, ASME, August, 1955, p. 963.

13 "What Happens When Hammer Hits Pile," by E. A. L. Smith, Engineering News-Record, McGraw-Hill Publishing Co., Inc., New York, N. Y., September 5, 1957, p. 46.

14 "Longitudinal Wave Transmission and Impact," by L. H. Donnell, Transactions, ASME, Vol. 52, 1930, p. 153.

15 "Theory of Elasticity," by S. Timoshenko and J. N. Goodier, McGraw-Hill Book Co., Inc., New York, N. Y., 2d Ed., 1951, p. 438.

16 "Stress Waves in Solids," by H. Kolsky, Oxford Univ. Press, London, New York, N. Y., 1953.

17 "Stress Wave Propagation in Rods and Beams," by H. N. Abramson, H. J. Plass, and E. A. Ripperger, Advances in Applied Mechanics, Vol. 5, Academic Press Inc., New York, N. Y., 1958, p. 111.

18 "Recent Developments in Elastic Wave Propagation," by Julius Miklowitz, Applied Mechanics Reviews, Vol. 13, No. 12, December, 1960, p. 865.

point resistance is accounted for by a single spring at the point of the pile. The characteristics of these various components will be described in greater detail.

Actual situations may deviate from that in Fig. 1. For example, a cushion block might not be used or an anvil may be placed between the ram and cap-block. Such cases are readily accommodated.

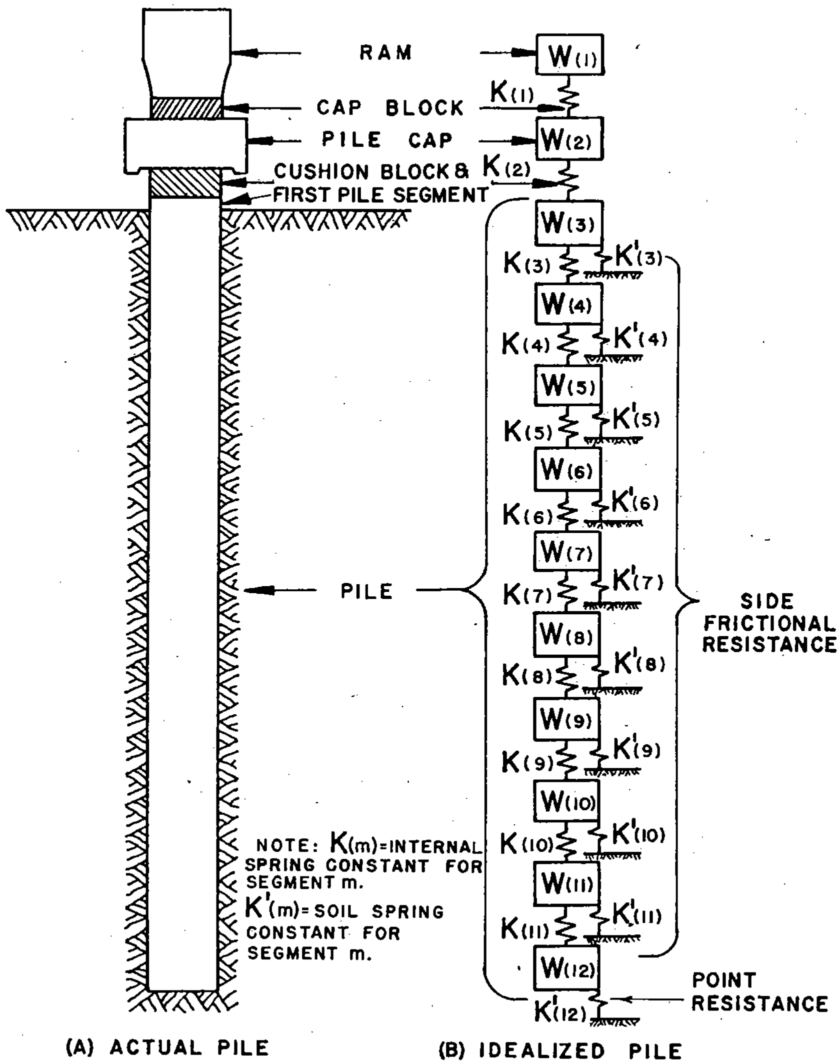

FIG. 1.—IDEALIZATION OF A PILE FOR PURPOSE OF ANALYSIS

Internal Springs.—The ram, capblock, pile cap, and cushion block may in general be considered to consist of "internal" springs, although in the representation of Fig. 1(b) the ram and the pile cap are treated as though rigid (a reasonable assumption for many practical cases).

Figs. 2(a), 2(b), 2(c), and 2(d) suggest different possibilities for representing the load-deformation characteristics of the internal springs. In Fig. 2(a) the material is considered to experience no internal damping. In Figs. 2(b) and 2(d), the material is assumed to have internal damping according to the linear relationships shown. In Fig. 2(c), loading and unloading are consid-

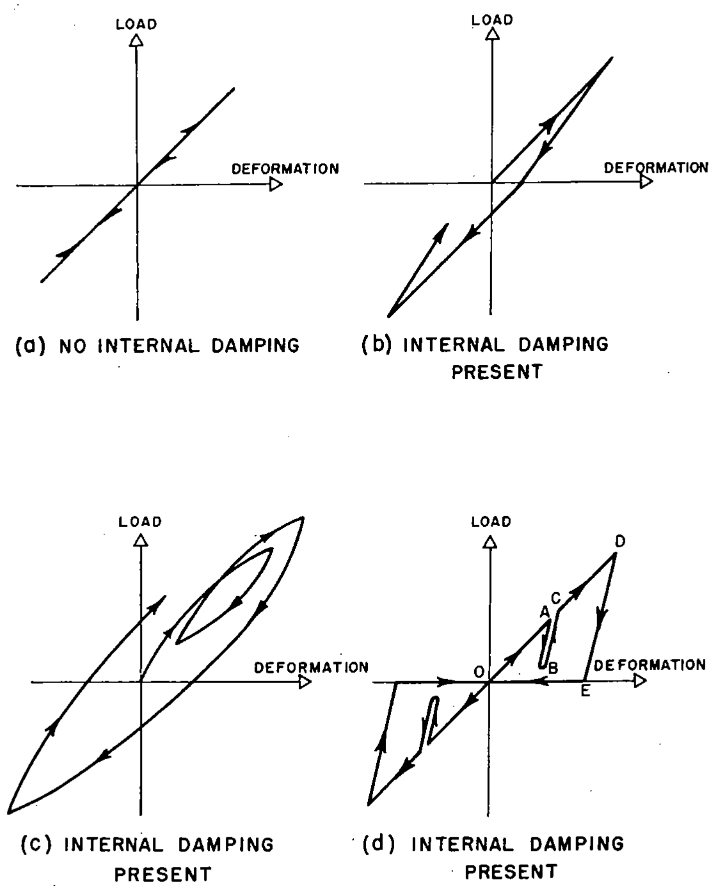

FIG. 2.—LOAD-DEFORMATION RELATIONSHIPS FOR INTERNAL SPRINGS

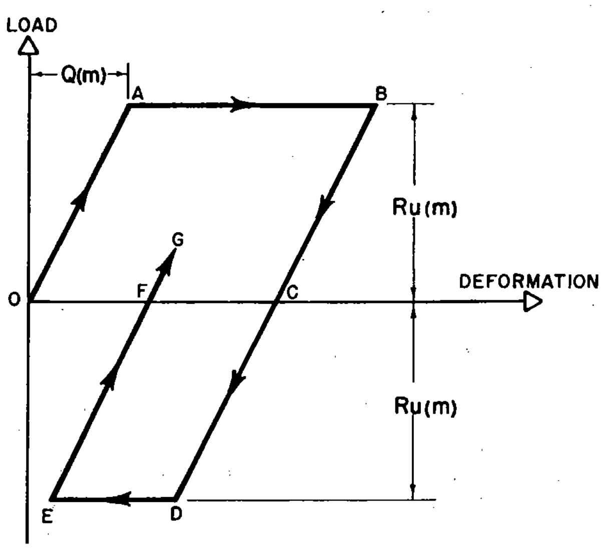

FIG. 3.—LOAD-DEFORMATION CHARACTERISTICS ASSUMED FOR SOIL SPRING M

ered to occur along a hysteresis loop. The assumption that should be made for a given problem depends on the material and its known load-deformation behavior under dynamic conditions. Future investigation will probably shed more light on this subject and will indicate which of Figs. 2(a), 2(b), 2(c), and 2(d) is most realistic for a given case. Furthermore, other representations can be incorporated if desired.

External Springs.—The resistance to dynamic loading afforded by the soil in shear along the outer surface of the pile and in bearing at the point of the pile is not clearly understood. Future studies will perhaps provide more information on this question. Fig. 3 shows the load-deformation characteristics assumed for the soil in Smith's procedure, exclusive of damping effects. The path OABCDEFG represents loading and unloading in side friction. In the present paper, both side-friction and point-bearing forces are permitted to act on the last pile segment. In this case, the point-bearing force must be prevented from exerting tension on the pile. Thus, loading and unloading would occur along OABCFCB. Smith[5] follows a somewhat different approach in combining point bearing and side friction.

It is seen that the characteristics of Fig. 3 are defined essentially by the quantities "Q" and "Ru." "Q" is termed the quake and represents the maximum deformation that may occur elastically. "Ru" is the ultimate ground resistance, or the load at which a spring behaves purely plastically.

A load-deformation diagram of the sort of Fig. 3 may be established separately for each spring. Thus,

$$K'(m) = \frac{Ru(m)}{Q(m)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in which K'(m) is the spring constant (during elastic deformation) for external spring m.

Basic Equations.—Eqs. 3 through 7 were developed by Smith[5]:

$$D(m,t) = D(m,t-1) + 12\Delta t\ V(m,t-1) \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$C(m,t) = D(m,t) - D(m+1,t) \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$F(m,t) = C(m,t)\ K(m) \quad \ldots\ldots\ldots\ldots\ldots \quad (5)$$

$$R(m,t) = \left[D(m,t) - D'(m,t)\right] K'(m) \left[1 + J(m)V(m,t-1)\right] \quad \ldots\ldots \quad (6)$$

$$V(m,t) = V(m,t-1) + \left[F(m,t) - F(m,t) - R(m,t)\right] \frac{g\ \Delta t}{W(m)} \quad \ldots\ldots \quad (7)$$

in which ( ) represents a functional designation; m denotes the element number; t denotes the time interval number; Δt is the size of time interval in seconds; C(m,t) represents the compression of internal spring m in time interval t, in inches; D(m,t) describes the displacement of element m in time interval t, in inches; D'(m,t) represents the plastic displacement of external spring m in time interval t, in inches; F(m,t) is the force in internal spring m in time interval t, in pounds; g represents the acceleration due to gravity, in feet per second squared; J(m) is the damping constant of soil at element m, in seconds per foot; K(m) is the spring constant associated with internal spring m, in pounds per inch; K'(m) is the spring constant associated with external spring m, in pounds per inch; R(m,t) is the force exerted by external spring m on element m in time interval t, in pounds; V(m,t) represents the velocity of element m in time interval t, in feet per second; and W(m) is the weight of element m, in pounds. This notation differs slightly from that used by Smith. Also, Smith restricts the soil damping constant J and the quake Q to two values each, one for the point of the pile in bearing and one for the side of the pile in friction. Although present knowledge of the damping behavior of soils perhaps does not justify greater refinement, J and Q are treated herein as functions of m for the sake of generality.

The use of a spring constant K(m) implies a load-deformation behavior of the kind shown in Fig. 2(a). For this situation, K(m) is the slope of the straight line. Smith develops special relationships to account for internal damping in the capblock and the cushion block. He obtains the following equation for alternate use with Eq. 5:

$$F(m,t) = \frac{K(m)}{[e(m)]^2} C(m,t) - \left\{ \frac{1}{[e(m)]^2} - 1 \right\} K(m)\, C(m,t)_{max} \quad \ldots \quad (8)$$

in which e(m) is the coefficient of restitution of internal spring m; and $C(m,t)_{max}$ denotes the temporary maximum value of C(m,t). With reference to Fig. 1, Eq. 8 would be applicable in the calculation of the forces in internal springs m = 1 and m = 2 in the ranges ABC and DE in Fig. 2(d). The load-deformation relationship characterized by Eqs. 5 and 8 is illustrated by the path OABCDEO. For a pile cap or a cushion block no tensile forces can exist; consequently, only this part of the diagram applies. Intermittent unloading-loading is typified by the path ABC, established by control of the quantity $C(m,t)_{max}$ in Eq. 8. The slope of lines AB, BC, and DE depends on the coefficient of restitution e(m).

The computations proceed as follows;

1. The initial velocity of the ram is determined from the properties of the pile driver. Other time-dependent quantities are initialized at zero.
2. Displacements D(m,1) are calculated by Eq. 3. It is to be noted that V(1,0) is the initial velocity of the ram.
3. Compressions C(m,1) are calculated by Eq. 4.
4. Internal spring forces F(m,1) are calculated by Eq. 5 or Eq. 8 as appropriate.
5. External spring forces R(m,1) are calculated by Eq. 6.
6. Velocities V(m,1) are calculated by Eq. 7.
7. The cycle is repeated for successive time intervals.

In Eq. 6 plastic deformation D'(m,t) for a given external spring follows Fig. 3 and may be determined by special routines. For example, when D(m,t) is less than Q(m), D'(m,t) is zero; when D(m,t) is greater than Q(m) along line AB (see Fig. 3), D'(m,t) is equal to D(m,t) - Q(m).

Smith notes that Eq. 6 produces no damping when D(m,t) - D'(m,t) becomes zero. He suggests an alternate equation to be used after D(m,t) first becomes equal to Q(m):

$$R(m,t) = \left[D(m,t) - D'(m,t)\right] K'(m) + J(m)\, K'(m)\, Q(m)\, V(m,t-1) \quad \ldots \quad (9)$$

Care must be used to satisfy conditions at the head and point of the pile. Consider Eq. 5. When m = p, in which p is number of the last element of the pile, K(p) must be set equal to zero since there is no F(p,t) force (see Fig. 1). Also, at the point of the pile, the soil spring must be prevented from exerting tension on the pile point. In applying Eq. 7 to the ram (m = 1), F(0,t) should be set equal to zero.

For the idealization of Fig. 1, it is apparent that the spring associated with K(2) represents both the cushion block and the top element of the pile. It may be obtained by the following equation:

$$\frac{1}{K(2)} = \frac{1}{K(2)_{cushion}} + \frac{1}{K(2)_{pile}} \quad \ldots\ldots\ldots\ldots \quad (10)$$

A more complete discussion of digital-computer programming details and recommended values for various physical quantities are given by Smith.[5]

It may be noted that the wave equation (Eq. 1) does not appear explicitly in the basic equations given for Smith's idealization. However, Eqs. 3-7 may be combined to produce a difference equation corresponding to Eq. 1 with resistance included.

APPLICATIONS AND EXTENSIONS OF THEORY

General.—The basic equations previously presented have been programmed for use as an analytical tool both in practical engineering applications and also in research. As a means of describing certain extensions to the procedure as well as illustrating its use in investigating the influence of various parameters, several situations have been studied. Fig. 4 identifies four different piles: piles I, III, and IV are prestressed concrete; pile II is a steel H-section. For each pile, different cases of soil resistance are considered.

Application to Ideal Pile; Pile I Studies.—For purposes of comparing Smith's procedure with the exact method of analyzing elastic bars, Pile I is considered. Two cases are treated: in the first, the point is free; in the second, the point is fixed. It is noteworthy that exact solutions are feasible only for pile problems involving special conditions of material properties and soil resistance. However, problems encountered in practice do not usually fall into this category.

The following information, in addition to that given in Fig. 4, is applicable to the Pile I studies: W(ram) = 11,500 lb; V(ram,0) = 14.45 ft per sec; K(cushion block) = 3,930,000 lb per in.; and e(cushion block) = 1.00.

Figs. 5 and 6 show stress versus time plots at the mid-length of the pile for free and fixed end conditions, respectively. Solutions have been obtained for the exact solution of the one-dimensional wave equation (Eq. 1) and for Smith's discrete-element method using ten, twenty, and forty pile segments. The term "exact" as used here refers to the direct solution of Eq. 1. The

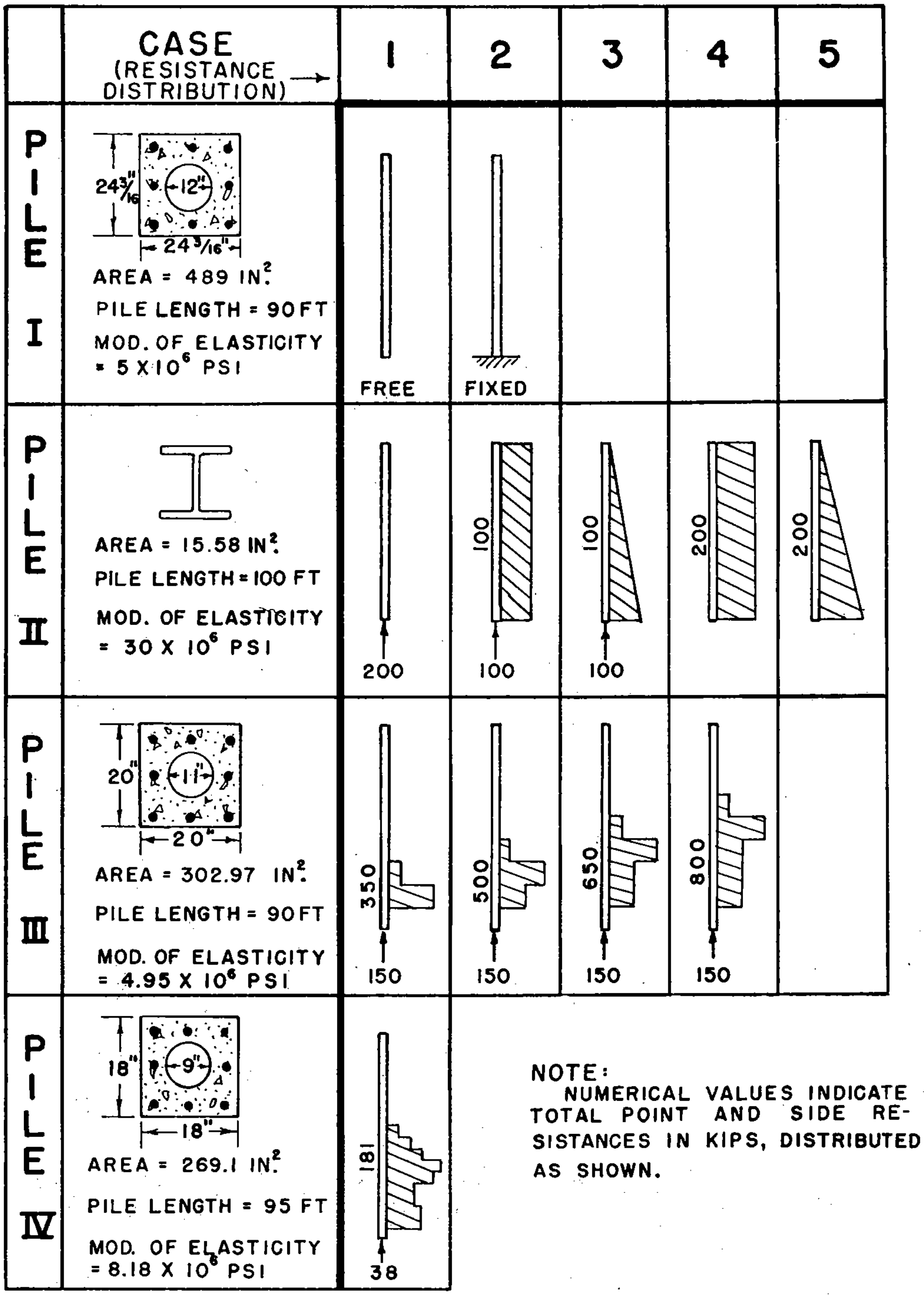

FIG. 4.—PILES AND ULTIMATE SOIL RESISTANCES USED IN STUDIES

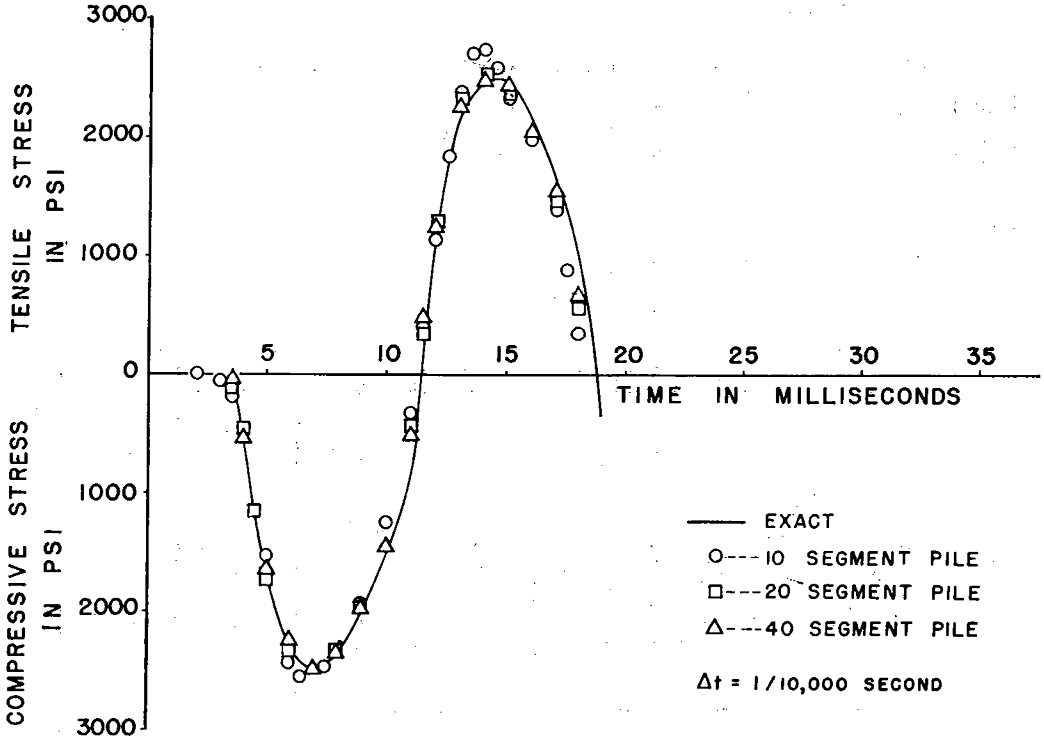

FIG. 5.—STRESS AT MID-LENGTH OF PILE I WITH END FREE

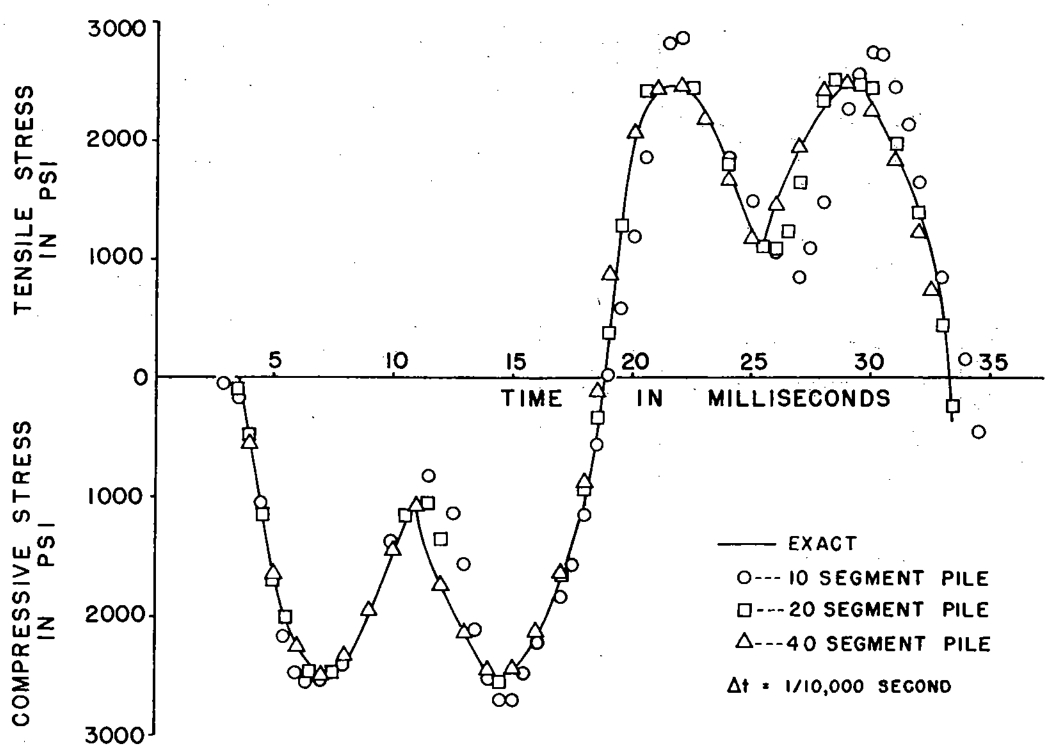

FIG. 6.—STRESS AT MID-LENGTH OF PILE I WITH END FIXED

value of Δt used for all Smith's solutions is 1/10,000 sec. It is interesting to note how increasing the number of pile segments improves the accuracy of the discrete-element solution, as might be expected.

The accuracy of the discrete-element solution is also related to the size of the time increment Δt. W. P. Heising,[19] in his discussion of the equation of motion for free longitudinal vibrations in a continuous elastic bar, points out that the discrete-element solution is an exact solution of the partial differential equation when

$$\Delta t = \frac{\Delta L}{\sqrt{\frac{E}{\rho}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

in which ΔL is the segment length. Smith[12] draws a similar conclusion. Figs. 7 and 8 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of the pile for various increments of time. These solutions use a segment length of 9 ft. If a time increment larger than that given by Eq. 11 is used, the discrete-element solution will diverge and no valid results can be obtained. As noted by Smith, in this case the numerical calculation of the discrete-element stress wave does not progress as rapidly as the actual stress wave. Consequently, the value of Δt given by Eq. 11 is called the "critical" value.

Heising[19] has also noted that when $\Delta t < \Delta L/\sqrt{E/\rho}$ is used in a discrete-element solution, a less accurate solution is obtained for the continuous bar. As Δt becomes progressively smaller, the solution approaches the actual behavior of the discrete-element system (segment lengths equal to ΔL) used to simulate the pile.

This in general leads to a less accurate solution for the longitudinal vibrations of a slender continuous bar. If, however, the discrete-element system were divided into a large number of segments, the behavior of this simulated pile would be essentially the same as that of the slender continuous bar irrespective of how small Δt becomes provided $\Delta L/\sqrt{E/\rho} \geqq \Delta t > 0$. This means that if the pile is divided into only a few segments, the accuracy of the solution will be more sensitive to the choice of Δt than if it is divided into many segments. For practical problems, a choice of Δt equal to approximately one-half the "critical" value appears suitable because inelastic springs and materials of different densities and elastic moduli are usually involved.

Figs. 9 and 10 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of Pile I (point free) for various time increments. Figs. 11 and 12 show comparisons of maximum tensile stress and maximum compressive stress, respectively, versus position along the length of Pile I (point fixed) for ten, twenty, and forty segments.

Effect of Gravity.—The procedure as presented by Smtih[5] does not account for the static weight of the pile. In other words, at $t = 0$ all springs, both internal and external, exert zero force. Stated symbolically, $F(m,0) = R(m,0) = 0$. If the effect of gravity is to be included, these forces must be given initial values to produce equilibrium of the system. Strictly speaking, these initial values should be those in effect as a result of the previous blow.

19 Discussion by W. P. Heising of "Impact and Longitudinal Wave Transmission," by E. A. L. Smith, Transactions, ASME, August, 1955, p. 963.

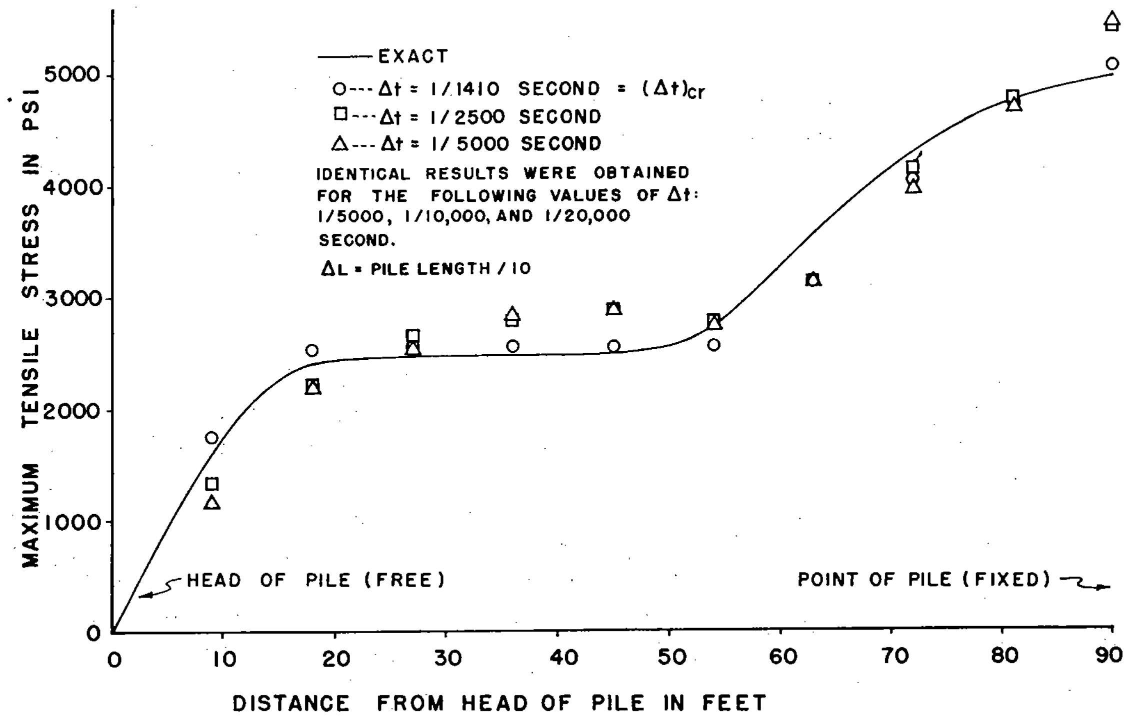

FIG. 7.—MAXIMUM TENSILE STRESS ALONG PILE I, CASE 2

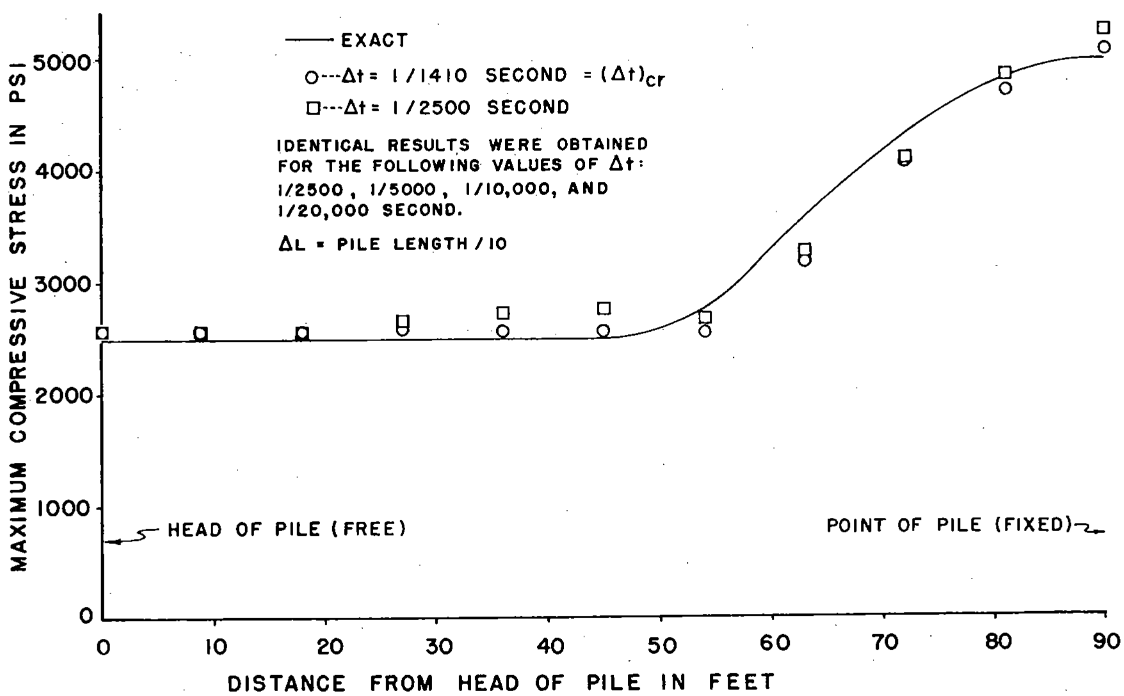

FIG. 8.—MAXIMUM COMPRESSIVE STRESS ALONG PILE I, CASE 2

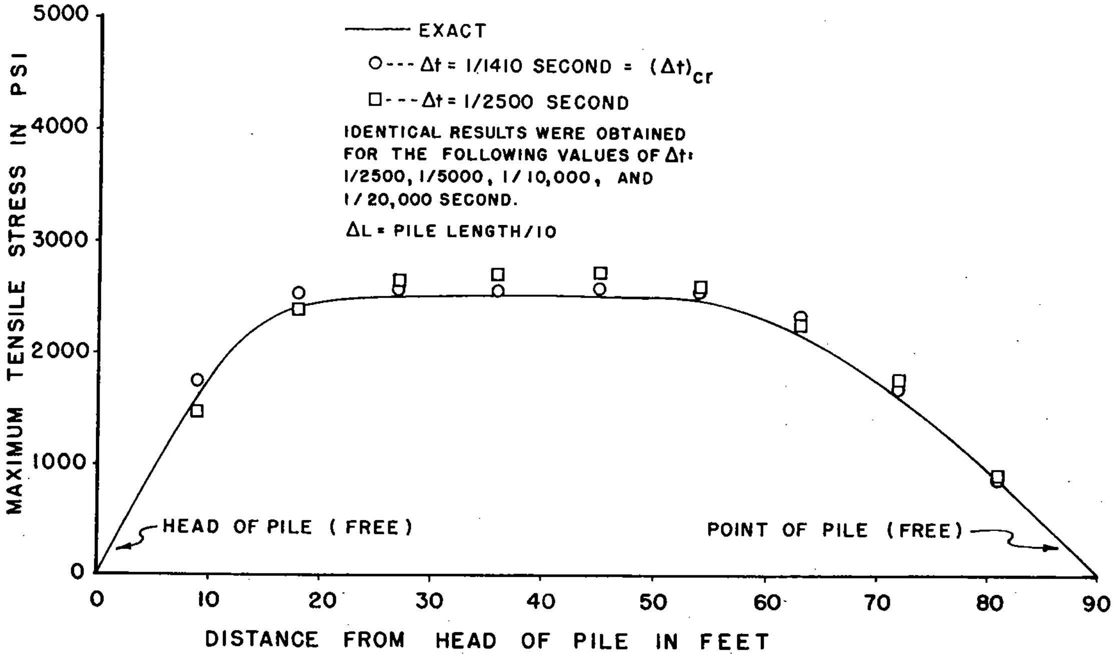

FIG. 9.—MAXIMUM TENSILE STRESS ALONG PILE I, CASE 1

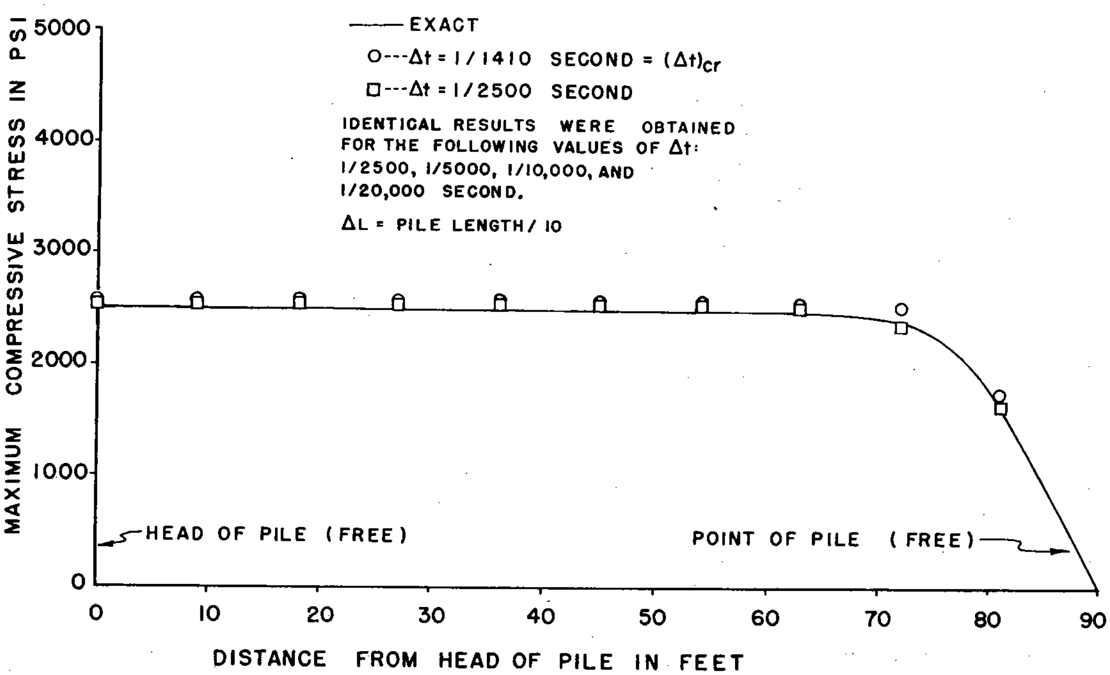

FIG. 10.—MAXIMUM COMPRESSIVE STRESS ALONG PILE I, CASE 1

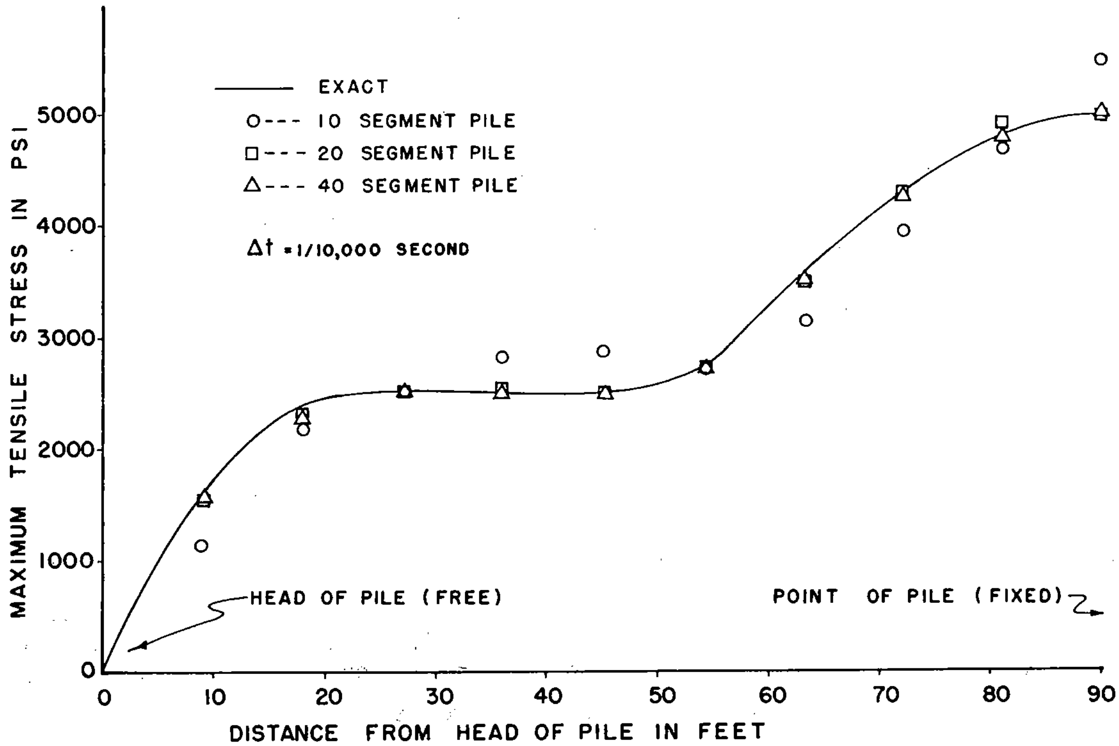

FIG. 11.—MAXIMUM TENSILE STRESS ALONG PILE I, CASE 2

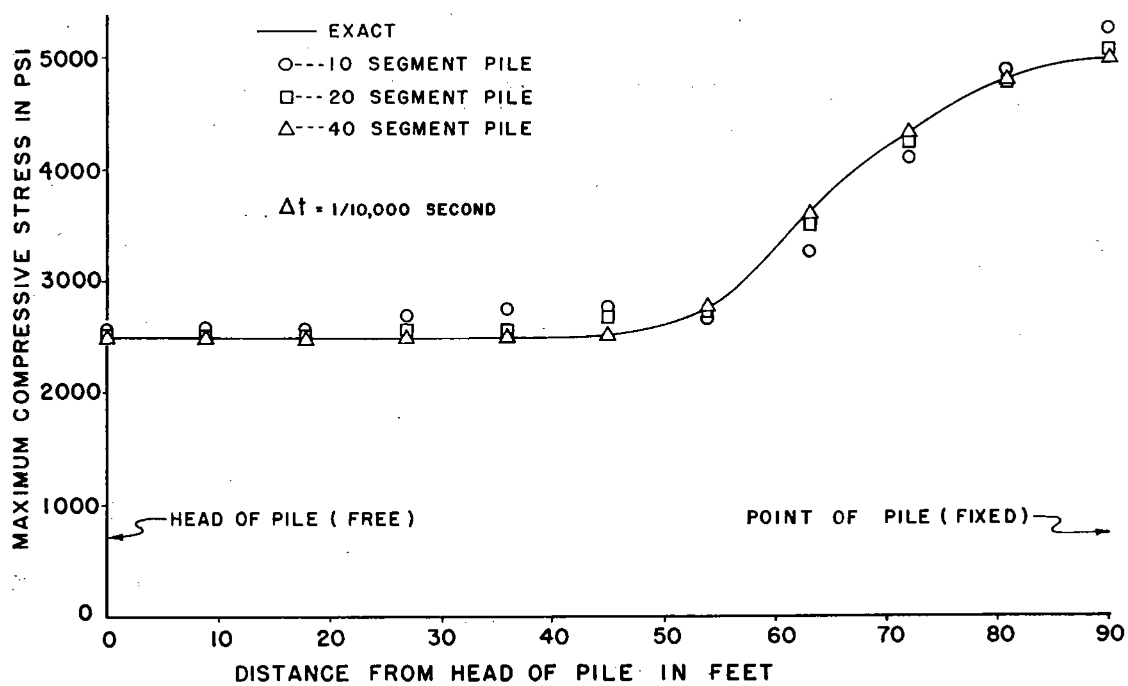

FIG. 12.—MAXIMUM COMPRESSIVE STRESS ALONG PILE I, CASE 2

Not only would it be awkward to "keep books" on the pile throughout the driving so as to identify the initial conditions for successive blows, but it is highly questionable that this refinement would be justified in light of other uncertainities that exist.

Two methods have been developed as a means of entering the gravity effect into the computations. The first of these has been proposed by Smith;[20] the second has been developed by the writers. Both will be described and compared.

Smith's Approximate Method.—Smith suggests that the external (soil) springs be assumed to resist the static weight of the system according to the relationship

$$R(m,0) = \left[\frac{Ru(m)}{Ru(total)}\right]\left[W(total)\right] \quad \ldots\ldots\ldots\ldots \quad (12)$$

in which W(total) is the total static weight resisted by soil, in pounds; and Ru(total) is the total ultimate ground resistance, in pounds. The quantity W(total) is found by

$$W(total) = W(b) + F(c) + \sum_{m=2}^{m=p} W(m) \quad \ldots\ldots\ldots\ldots \quad (13)$$

in which W(b) is the weight of the body of the hammer, excluding ram, in pounds; and F(c) is the force exerted by compressed gases, as under the ram of a diesel hammer, in pounds. The internal forces that initially exist in the pile may now be obtained:

$$F(1,0) = W(b) + F(c) \quad \ldots\ldots\ldots\ldots \quad (14)$$

and in general,

$$F(m,0) = F(m-1,0) + W(m) - R(m,0) \quad \ldots\ldots\ldots\ldots \quad (15)$$

In the absence of compressed gases and hammer weight resting on the pile system, the right side of Eq. 14 is zero. The amount that each internal spring m is compressed may now be expressed as

$$C(m,0) = \frac{F(m,0)}{K(m)} \quad \ldots\ldots\ldots\ldots \quad (16)$$

The displacement of the point may be obtained from

$$D(p,0) = \frac{R(p,0)}{K'(p)} \quad \ldots\ldots\ldots\ldots \quad (17)$$

By working progressively upward from the point, other displacements are determined from

$$D(m,0) = D(m+1,0) + C(m,0) \quad \ldots\ldots\ldots\ldots \quad (18)$$

[20] Personal correspondence dated April 13, 1961, from E. A. L. Smith to Charles H. Samson, Jr.

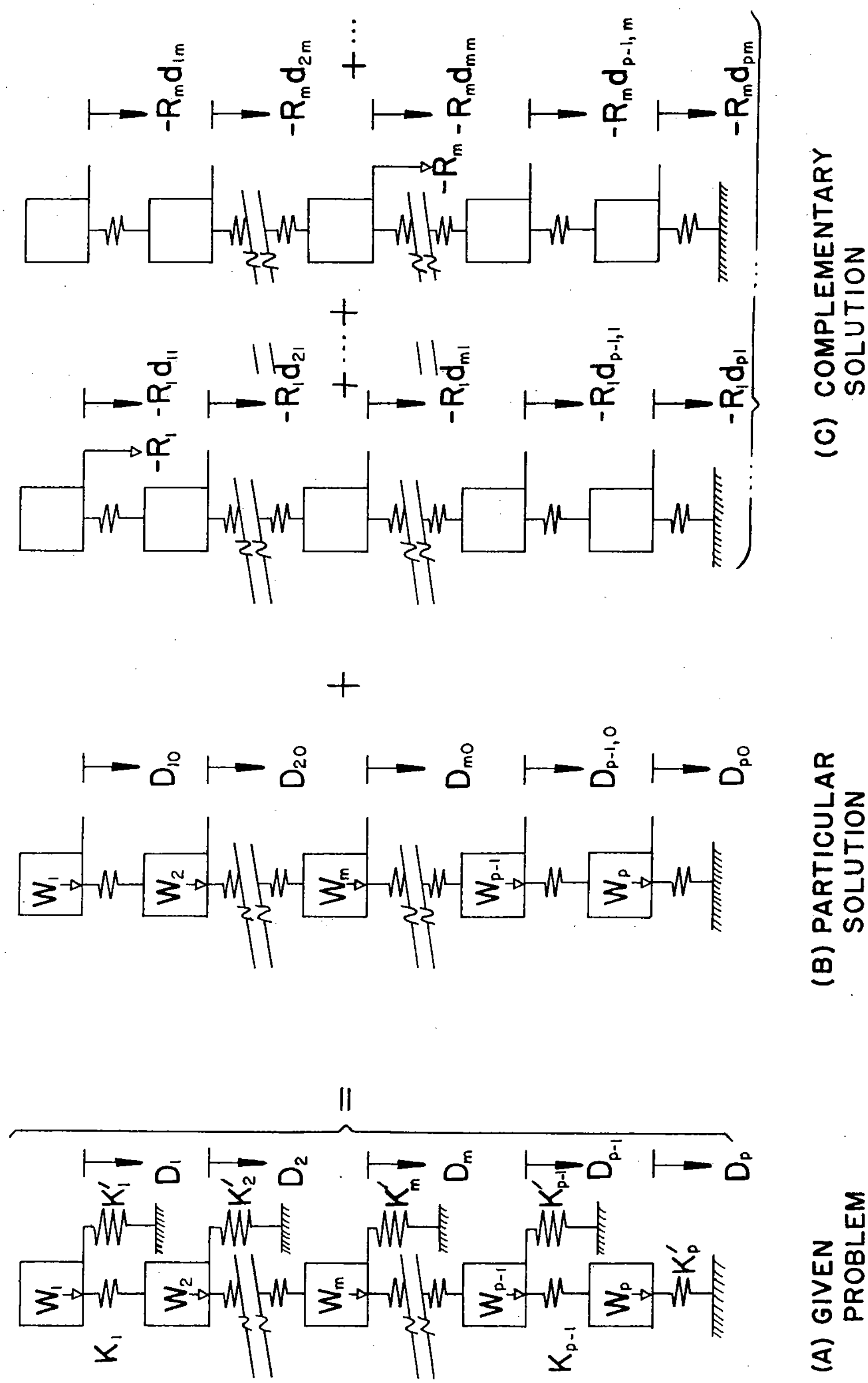

FIG. 13.—ILLUSTRATION OF INFLUENCE-COEFFICIENT METHOD OF SOLUTION

For the inclusion of the gravity effect, Eq. 7 should be modified as follows:

$$V(m,t) = V(m,t-1) + \left[F(m-1,t) - F(m,t) - R(m,t) + W(m)\right] \frac{g\,\Delta t}{W(m)} \quad \ldots \quad (19)$$

In order that the initial conditions of the external springs be compatible with the assumed initial forces R(m,0) and initial displacements D(m,0), plastic displacements D'(m,0) should be set equal to D(m,0) - R(m,0)/K'(m).

Influence-coefficient Methods.—A second approach that may be used is to treat the system as a statically indeterminate structure. The assumption is made that all springs behave as though linearly elastic in supporting the static weight. The influence-coefficient method[21] will be determined here as one means of solving the problem. Fig. 13 illustrates the two parts to the solution. The particular solution is obtained for the statically determinate structure (primary structure) formed by cutting all external springs except the last and by loading with the applied loads W_1, W_2, etc. In this particular discussion, subscripts will be used as a convenience in place of the functional designation used in other sections of the paper. The displacements for the various elements of this loaded structure are denoted by D_{10}, D_{20}, . . . D_{m0}, . . . and D_{p0}. In the formulation of the complementary solution, influence coefficients d_{11}, d_{12} . . . d_{pp} are generated by successively applying unit loads (positive downward) at points 1, 2, . . . p on the primary structure. For example, d_{12} is the deflection at point 1 caused by a unit load at point 2. Corresponding displacements caused by R_1, R_2, . . . R_p can now be expressed in terms of influence coefficients. For example, the complete equation for the actual displacement at a point m may be stated as

$$D_m = D_{m0} - d_{m1}R_1 - d_{m2}R_2 \ldots - d_{mm}R_m \ldots - d_{mp}R_p \quad \ldots \quad (20)$$

By considering the external spring at m as a free body, the following is obtained:

$$D_m = \frac{Rm}{K'_m} \quad \ldots\ldots\ldots\ldots \quad (21)$$

Substituting from Eq. 21 into Eq. 20 produces

$$\frac{R_m}{K'_m} = D_{m0} - d_{m1}R_1 - d_{m2}R_2 \ldots - d_{mm}R_m \ldots - d_{mp}R_p \quad \ldots \quad (22)$$

Similar equations may be obtained for the other points of redundancy.

Using matrix notation for a concise representation of the series of equations typified by Eq. 22 yields

$$\left([d] + \left[\frac{1}{K'}\right]\right)\{R\} = \{D_0\} \quad \ldots\ldots\ldots \quad (23)$$

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21 "Frame Analysis," by Arthur S. Hall and Ronald W. Woodhead, John Wiley & Sons, Inc., New York, N. Y., 1961.

in which

$$[d] = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ d_{21} & d_{22} & \cdots & d_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ d_{p1} & d_{p2} & \cdots & d_{pp} \end{bmatrix} \qquad (24)$$

$$\left[\frac{1}{K'}\right] = \begin{bmatrix} \frac{1}{K'_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{K'_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \frac{1}{K'_p} \end{bmatrix} \qquad (25)$$

$$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_p \end{Bmatrix} \qquad (26)$$

and

$$\{D_0\} = \begin{Bmatrix} D_{10} \\ D_{20} \\ \cdot \\ \cdot \\ \cdot \\ D_{p0} \end{Bmatrix} \qquad (27)$$

Considering that $[d]$, $[1/K']$, and $\{D0\}$ are known, the solution for $\{R\}$ may be obtained by premultiplying both sides of Eq. 23 by the inverse of $([d] + [1/K'])$;

thus,

$$\{R\} = \left([d] + \left[\frac{1}{K'}\right]\right)^{-1} D_0 \quad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

Of course, any method for solving the set of simultaneous equations implicit in Eq. 23 is satisfactory. However, the matrix formulation of Eq. 28 is quite suitable for digital-computer solution.

The solutions for $\{D_0\}$ and [d] are both associated with a statically determinate system [see Figs. 13(b) and 13(c)]. They may be found by the following expressions:

$$D_{p0} = \frac{1}{K'_p} \sum_{i=1}^{i=p} W_i \quad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

$$\underset{m \neq p}{D_{m0}} = D_{m+1,0} + \frac{1}{K_m} \sum_{i=1}^{i=m} W_i \quad \ldots\ldots\ldots\ldots \quad (30)$$

$$d_{mn} = D_{m0} \text{ in Eq. 30 with } W_i = \begin{cases} 1 \text{ for } i = n \\ 0 \text{ for } i \neq n \end{cases} \quad \ldots\ldots\ldots \quad (31)$$

It is seen that computations may proceed from the point upward for all displacement quantities for the particular and complementary solutions. For simplification, i is considered to begin with i = 1. This, of course, must be adjusted to a particular pile problem to exclude the ram and any other elements not having side springs.

At this stage, returning to the previous notation, all R(m,0) values may be established by Eq. 28. Furthermore, by Eq. 21 all D(m,0) values may be obtained. Compressions C(m,0) may be determined by

$$C(m,0) = D(m+1,0) - D(m,0) \qquad m \neq p \quad \ldots\ldots\ldots\ldots \quad (32)$$

Forces F(m,0) are now found from

$$F(m,0) = C(m,0)K(m) \qquad m \neq p \quad \ldots\ldots\ldots\ldots \quad (33)$$

Eq. 19 again provides the modified general velocity equation and computations may now proceed in the usual manner.

Pile II Studies.—Pile II is identified in Fig. 4. These studies are concerned with the effects of gravity, time interval magnitude, and segment length. All calculations associated with this pile were performed on the basis of the following given data: Q(m) = 0.1 in. for all m; J(m) = 0.05 sec per ft for $m \neq p$; J(p) = 0.15 sec per ft; W(ram) = 5,000 lb; W(pile cap) = 700 lb; K(capblock) = 2×10^6 lb per in.; e(capblock) = 0.50; and V(1,0) = 12.4 ft per sec. A special pile point weighing 100 lb is assumed. Various cases of loading are identified in Fig. 4. A total ground resistance, Ru(total, of 200,000 lb) is used in each case. Other details concerning Pile II are given in Fig. 4.

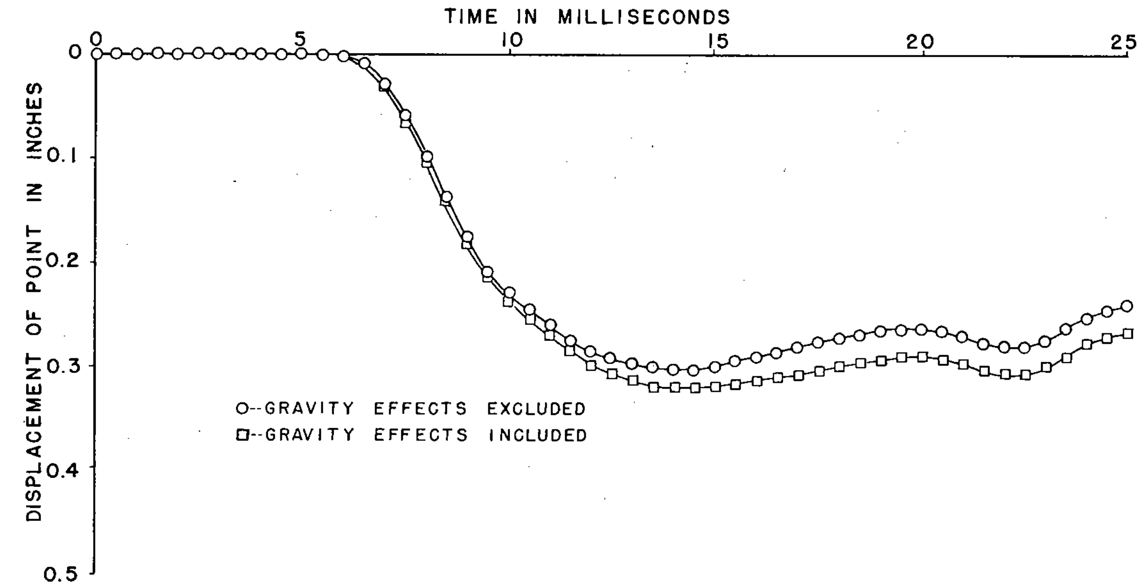

FIG. 14.—VARIATION OF POINT DISPLACEMENT WITH TIME: PILE II, CASE 1

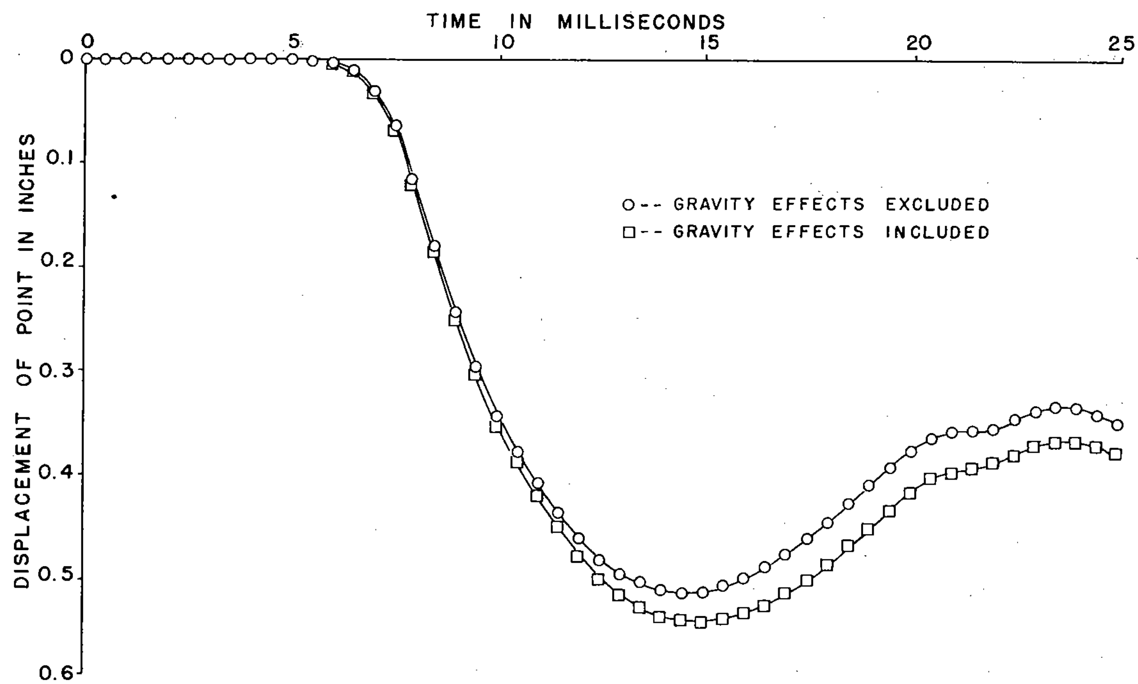

FIG. 15.—VARIATION OF POINT DISPLACEMENT WITH TIME: PILE II, CASE 4

Fig. 14 is a plot of point displacement versus time for Pile II, Case 1. The upper curve represents the solution with the gravity effect ignored. The lower curve represents the solution including gravity effects obtained both by Smith's approximate method and by the influence-coefficient method; the differences could not be plotted. Fig. 15 is a similar plot for Pile II, Case 4. The greatest effects of gravity are in the range of 5% to 10% for the plots shown.

Figs. 16 and 17 are plots of midpoint stress versus time for Pile II, Cases 1 and 4, respectively. The gravity effect on stresses for these two cases is small. It is of interest to note the comparison of R(m,0) values used in considering Pile II, Case 4. Table 1 shows these values for gravity excluded, for Smith's approximate solution and for the influence-coefficient solution.

Table 2 shows the effect of gravity on maximum compressive force, maximum tensile force, and the permanent set for all five cases of soil resistance for Pile II. The solutions with gravity included were obtained by the influence-coefficient method; however, they are quite similar to those found by Smith's approximate method.

TABLE 1.—R(M,0) VALUES

m	R(m,0), in pounds		
	Gravity excluded	Smith's approximation	Influence-coefficient solution
3	0	610	679.91
4	0	610	651.70
5	0	610	629.73
6	0	610	612.87
7	0	610	600.27
8	0	610	591.26
9	0	610	585.41
10	0	610	582.39
11	0	610	582.06
12	0	610	584.40

Table 3 provides a comparison of maximum compressive forces and permanent set for various magnitudes of time intervals. For this case, the variation in results over the range of time intervals considered is small. For Pile II, the $(\Delta t)_{cr}$ for a segment length of 10 ft (ten segments) is 1/1,667 sec. All results thus far discussed for Pile II were obtained using a segment length of 10 ft.

Table 4 provides a comparison of maximum compressive forces at different positions along the pile for different magnitudes of time intervals and different segment lengths. The results apply to Pile II, Case 2. It is seen that the forces compare closely at a given point on the pile.

Pile III Studies.—The different resistance cases considered for Pile III are defined in Fig. 4. These studies illustrate the influence of pile driver, cushion block, and pile characteristics on pile behavior. The following data are used:

Ram Weight.—W(A) = 4,850 lb; W(B) = 9,300 lb; and W(C) = 14,000 lb. In the case of ram A, two possibilities are examined: in the first it is assumed that no explosive pressure exists beneath the ram; in the second it is assumed that an explosive pressure of 158,700 lb is present. This simulates a pressure that might be encountered in certain types of diesel hammers.

Capblock.—K(capblock) = 240 x 10^6 lb per in. and e(capblock) = 0.50.

Pile Cap.—One pile is considered for all solutions. It is assumed to be rigid with W(pile cap) = 1,150 lb.

Cushion Block.—Two cushion blocks are considered, both assumed weightless. Stiffnesses are specified as follows: K(A) = 16.7 x 10^6 lb per in.; K(B) = 167 x 10^6 lb per in.; and e(cushion block) = 1.00.

Pile.—ΔL = 10 ft for each of 9 segments. Unless otherwise specified, a modulus of elasticity of 4.95 x 10^6 psi is assumed.

Soil.—J(m) = 0.05 sec per ft for $m \neq p$; J(p) = 0.15 sec per ft; and Q(m) = 0.10 in. for all m.

Figs. 18, 19, 20, and 21 show plots of maximum tensile stress developed in the pile versus ram velocity for different ram weights and different cushion blocks. The results for ram A are given with and without explosive pressure. Similar plots of maximum compressive stress developed in the pile versus ram velocity are provided in Figs. 22, 23, 24, and 25.

Although this study is of limited scope, it illustrates a way in which generalized information can be presented. It is also of interest to appraise the particular results obtained for the ranges of variables treated.

For a given ram, as might be expected, both tensile and compressive stresses usually increase with velocity.

In most cases, the effect of the stiffer cushion block is to produce higher tensile stresses. Minor exceptions may be noted for ram C in Figs. 18, 19, and 20. The influence of cushion-block stiffness on development of tensile stress is quite pronounced in some cases. For example, the stiffer cushion block causes 59% greater tension velocity of 17.8 ft per sec, and Case 1 soil resistance.

The effect of explosive pressure for the ranges of variables considered does not appear marked. For example, for the Case 1 soil resistance, the soft cushion block, and ram velocity of 17.8 ft per sec, the explosive pressure produces a decrease of 7% in the maximum tensile stress developed.

In order to gain information concerning the effect of modulus of elasticity of the pile material, the combination of Case 1 soil resistance, ram A with no explosive pressure, and stiff and soft cushion blocks are considered for moduli of elasticity of 2.5 x 10^6 psi and 7.5 x 10^6 psi, in addition to the value of 4.95 x 10^6 psi used in previous calculations. Figs. 26 and 27 illustrate the results in the form of maximum tensile stress and maximum compressive stress, respectively, versus ram velocity. At all velocities considered, both tensile and compressive stresses increase with increase in modulus of elasticity of the pile material.

Comparison of Theoretical and Field Test Results; Pile IV Studies.—Pile IV was field tested during the construction of the Nueces Bay Causeway at Corpus Christi, Tex. Stresses were recorded on a high-speed recording oscillograph. The variation of stress with time at a point 9.5 ft from the head of the pile is plotted in Figs. 28, 29, and 30 as a basis for comparison with theory. The pile had penetrated 45 ft into a soft marine silty-clay.

The theoretical solutions for all three figures were obtained with the following specified information: W(ram) = 4,850 lb and V(ram,0) = 13.8 ft per sec. The gravity effect of the pile was neglected.

The differences in the theoretical solutions result from the manner of accounting for internal damping in the pile. In the theoretical solution of Fig. 28, the pile material is considered to be perfectly elastic. In the solution

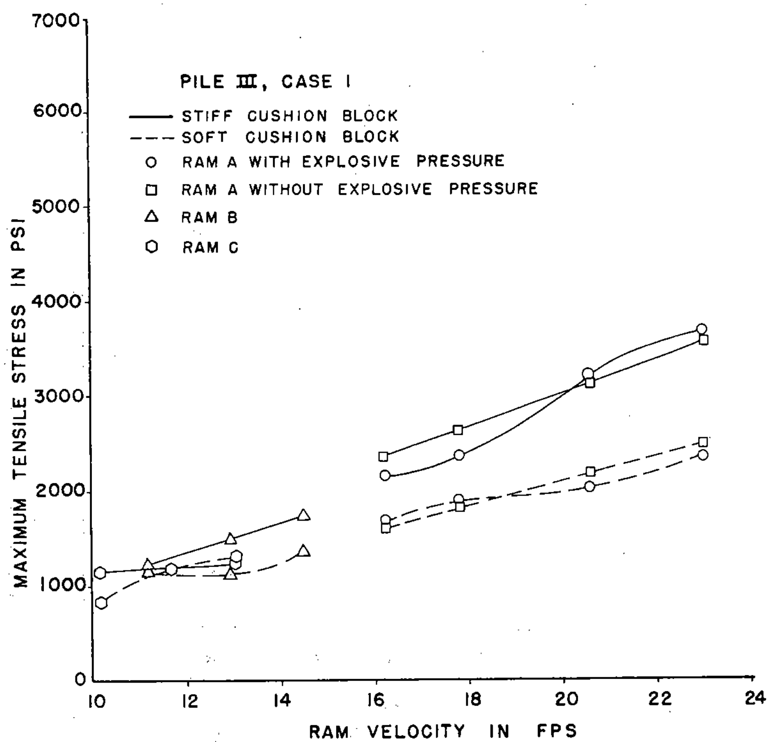

FIG. 18.—VARIATION OF MAXIMUM TENSILE STRESS WITH RAM VELOCITY: PILE III, CASE 1

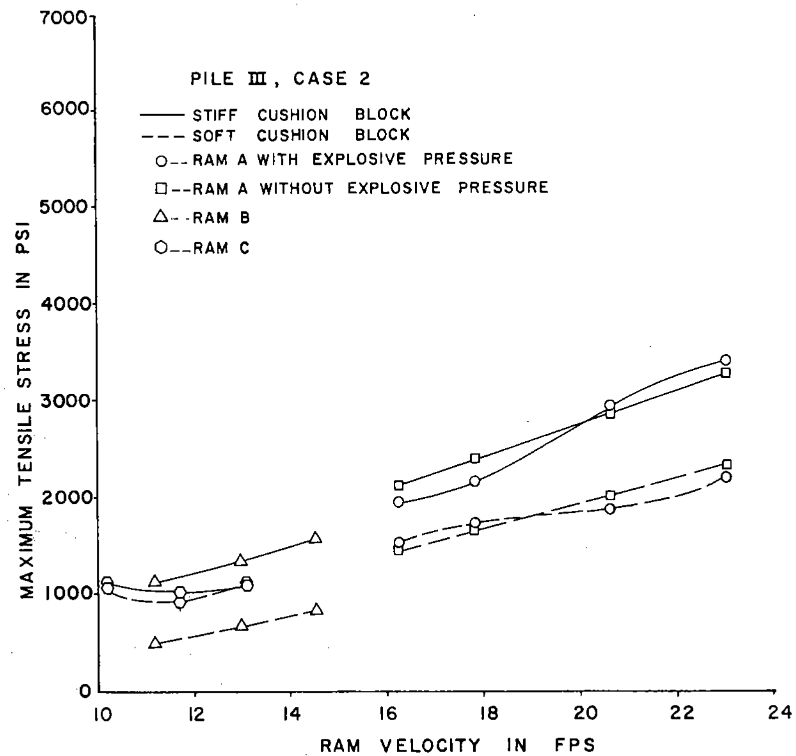

FIG. 19.—VARIATION OF MAXIMUM TENSILE STRESS WITH RAM VELOCITY: PILE III, CASE 2

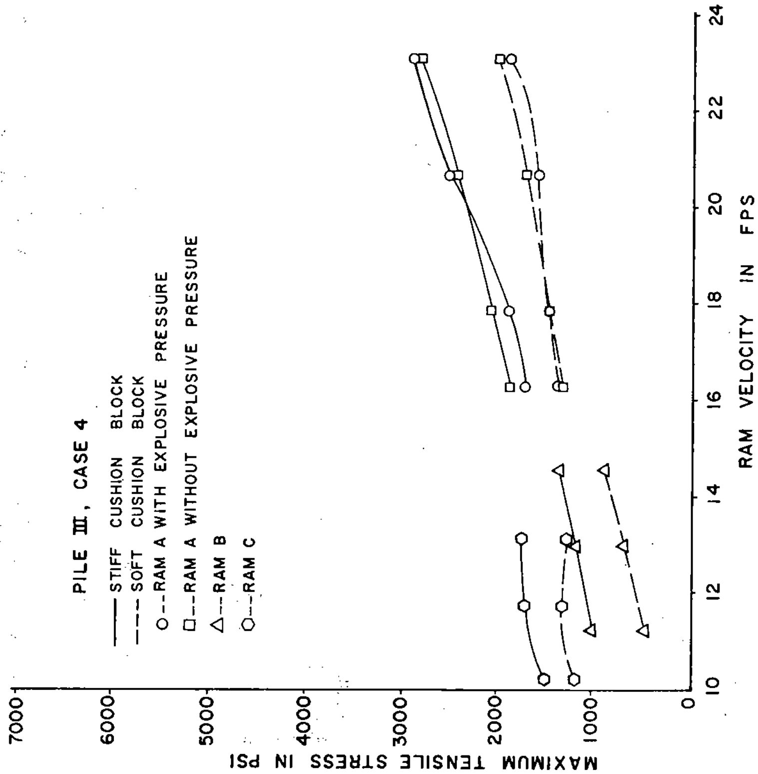

FIG. 21.—VARIATION OF MAXIMUM TENSILE STRESS WITH RAM VELOCITY: PILE III, CASE 4

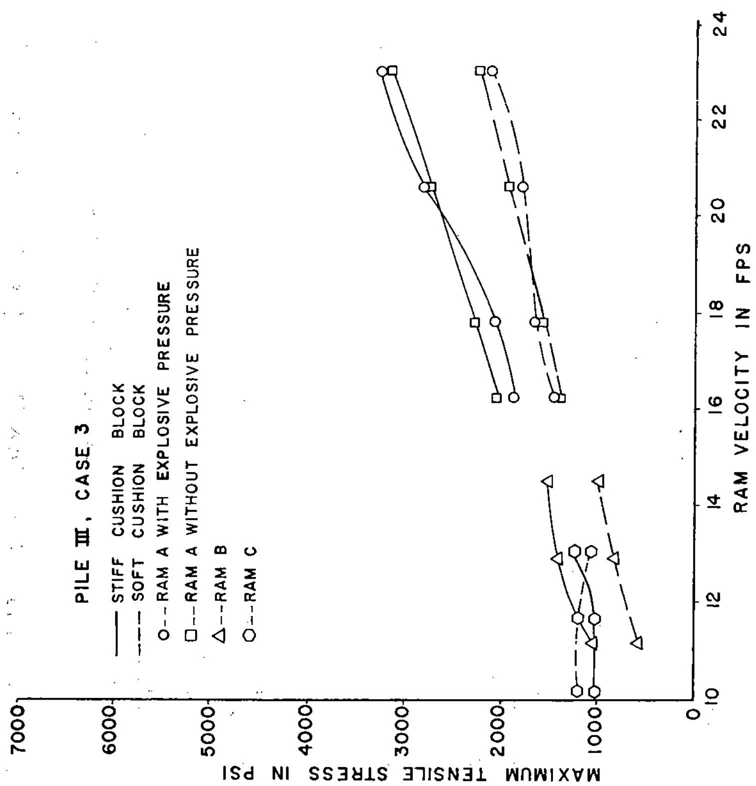

FIG. 20.—VARIATION OF MAXIMUM TENSILE STRESS WITH RAM VELOCITY PILE III, CASE 3

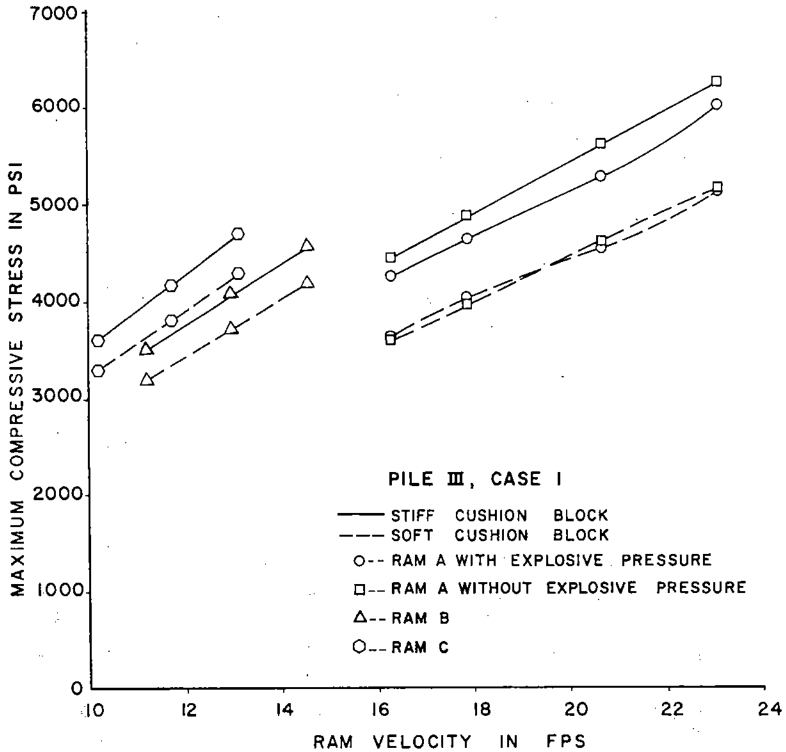

FIG. 22.—VARIATION OF MAXIMUM COMPRESSIVE STRESS WITH RAM VELOCITY: PILE III, CASE 1

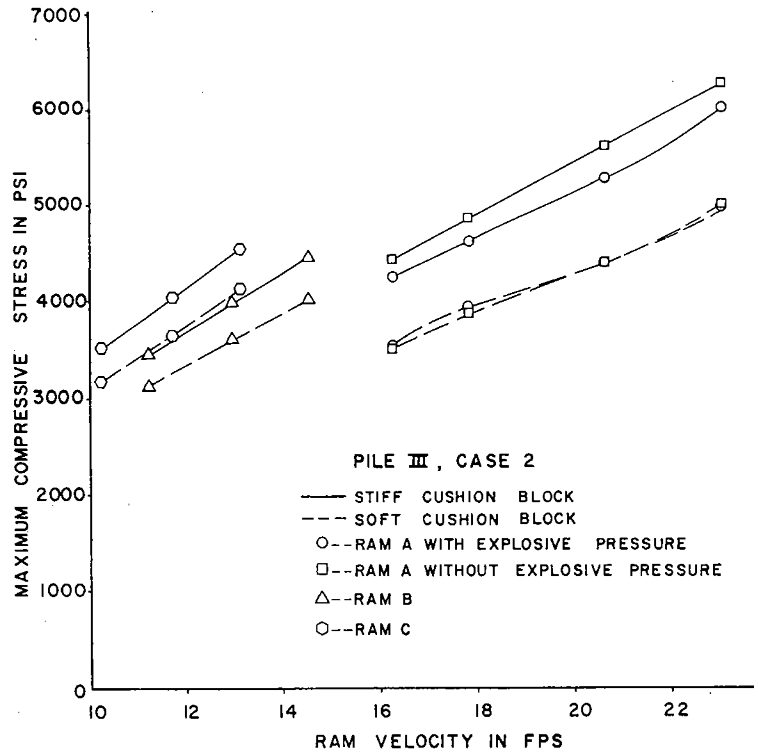

FIG. 23.—VARIATION OF MAXIMUM COMPRESSIVE STRESS WITH RAM VELOCITY: PILE III, CASE 2

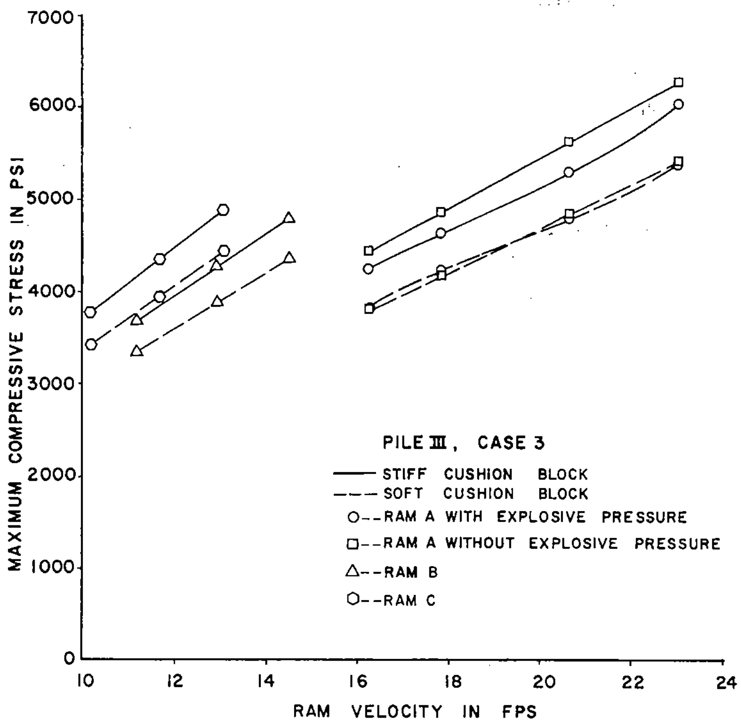

FIG. 24.—VARIATION OF MAXIMUM COMPRESSIVE STRESS WITH RAM VELOCITY: PILE III, CASE 3

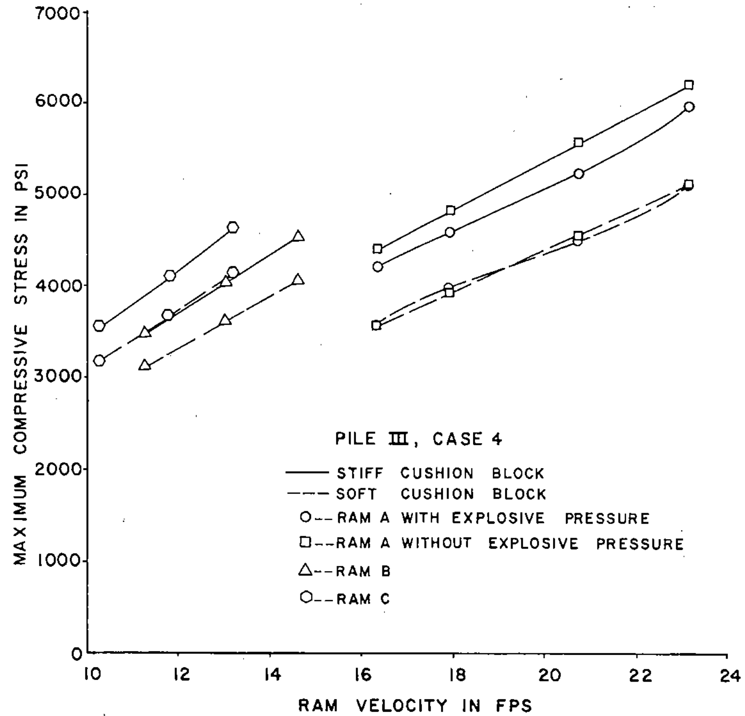

FIG. 25.—VARIATION OF MAXIMUM COMPRESSIVE STRESS WITH RAM VELOCITY: PILE III, CASE 4

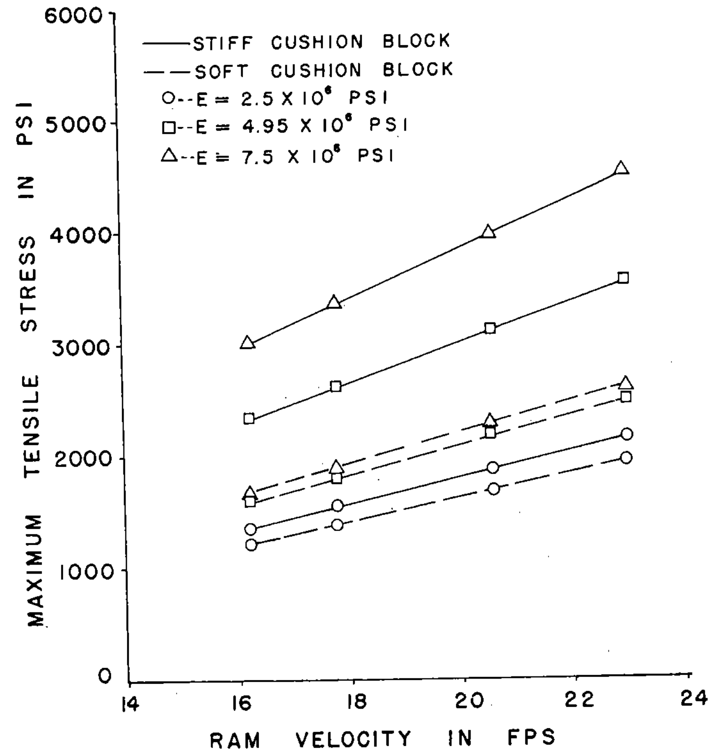

FIG. 26.—INFLUENCE OF MODULUS OF ELASTICITY ON MAXIMUM TENSILE STRESS: PILE III, CASE 1

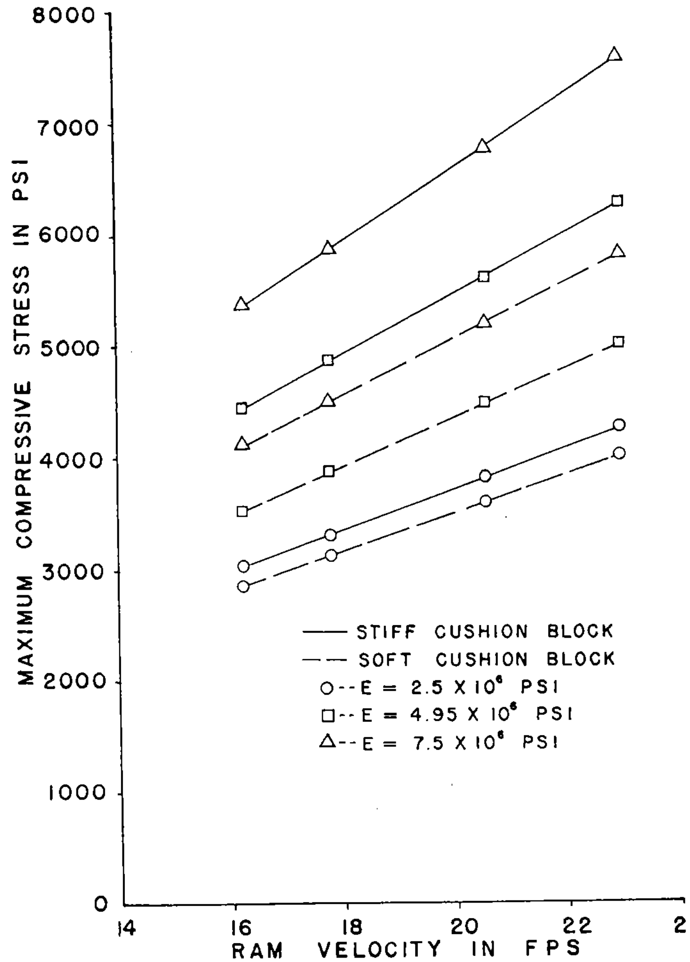

FIG. 27.—INFLUENCE OF MODULUS OF ELASTICITY ON MAXIMUM COMPRESSIVE STRESS: PILE III, CASE 1

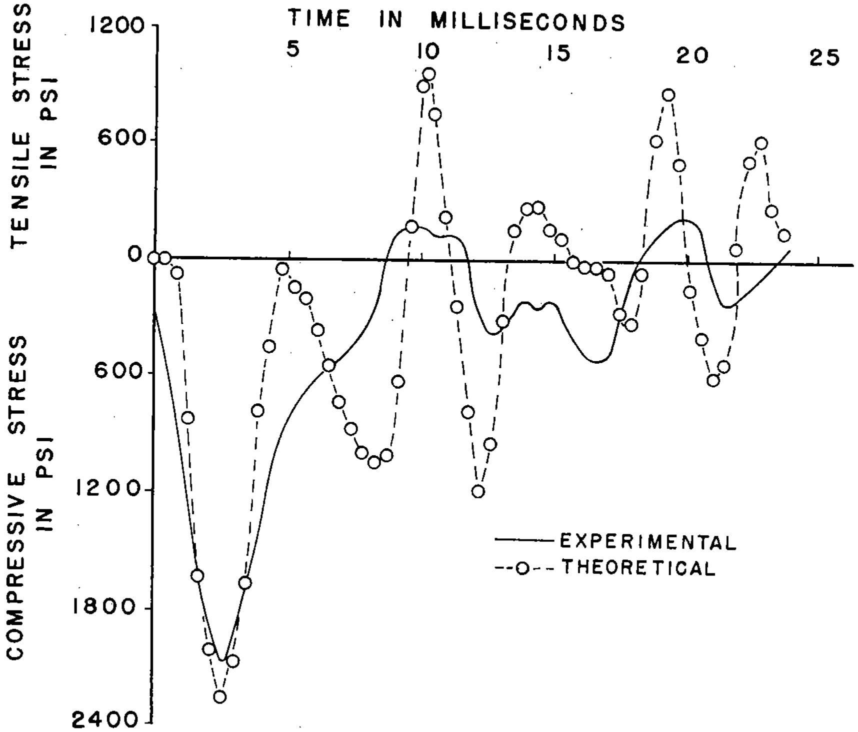

FIG. 28.—COMPARISON OF THEORETICAL AND TEST RESULTS: PILE IV WITH NO INTERNAL DAMPING (e = 1.0 AND B = 0.0)

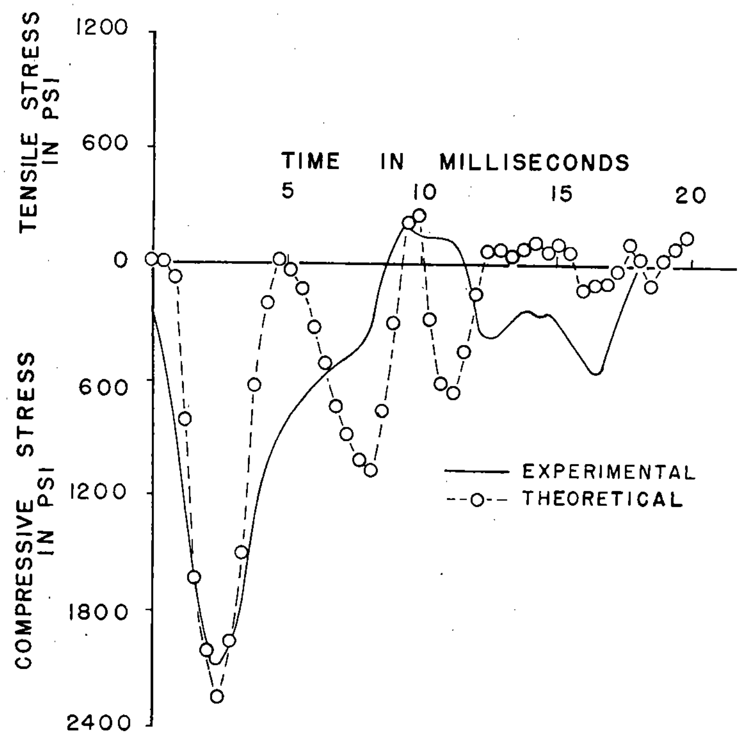

FIG. 29.—COMPARISON OF THEORETICAL AND TEST RESULTS: PILE IV WITH INTERNAL DAMPING (e = 0.8)

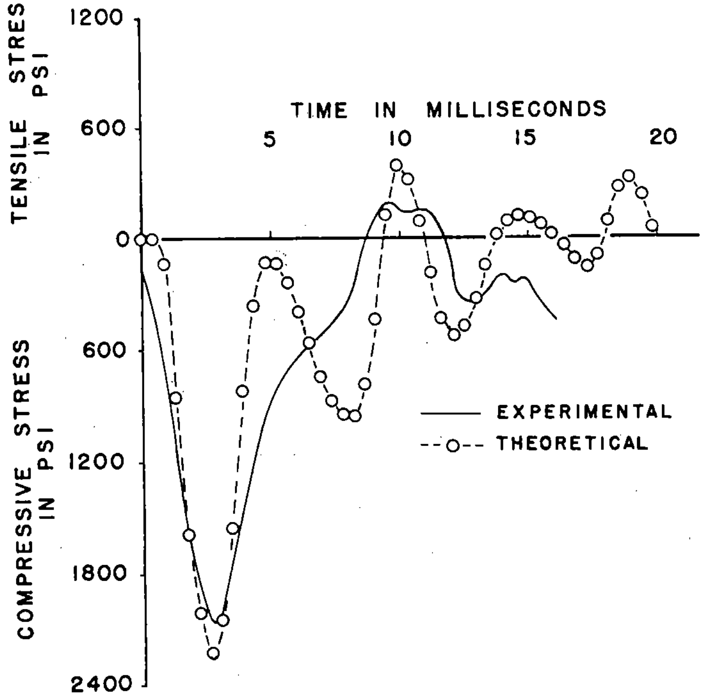

FIG. 30.—COMPARISON OF THEORETICAL AND TEST RESULTS: PILE IV WITH INTERNAL DAMPING (B = 0.0016)

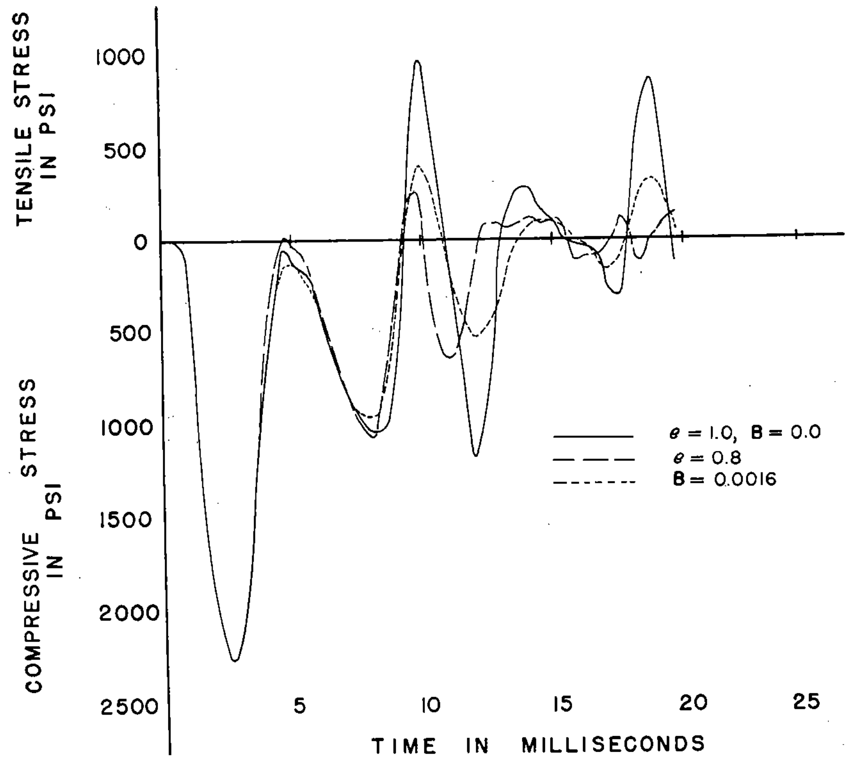

FIG. 31.—COMPARISON OF THEORETICAL CURVES FROM FIGS. 28, 29, AND 30

shown in Fig. 29, a load-deformation relationship of the form of Fig. 2(b) is assumed in which the coefficient of restitution is 0.8. Although there is no particular justification for this specific shape, it does serve to provide a different internal load-deformation behavior by which internal damping occurs.

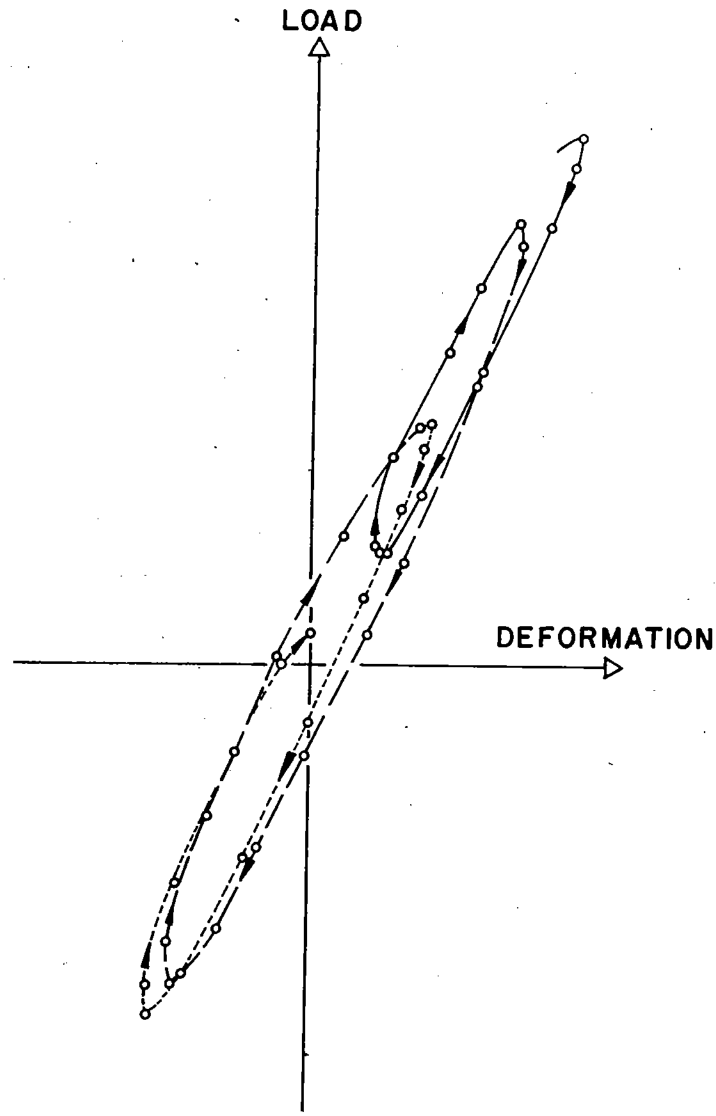

FIG. 32.—SEGMENT OF A TYPICAL LOAD-DEFORMATION CURVE OBTAINED BY EQ. 34

Smith[5] has proposed another means of accounting for internal damping. In place of Eq. 5 Smith suggests

$$F(m,t) = C(m,t)K(m) + B(m)K(m)\,\frac{C(m,t) - C(m,t-1)}{12\,\Delta t} \quad \ldots\ldots \quad (34)$$

in which B(m) is a damping constant for internal spring m. Smith states that a value of B of approximately 0.0002 in. sec per ft will produce a narrow

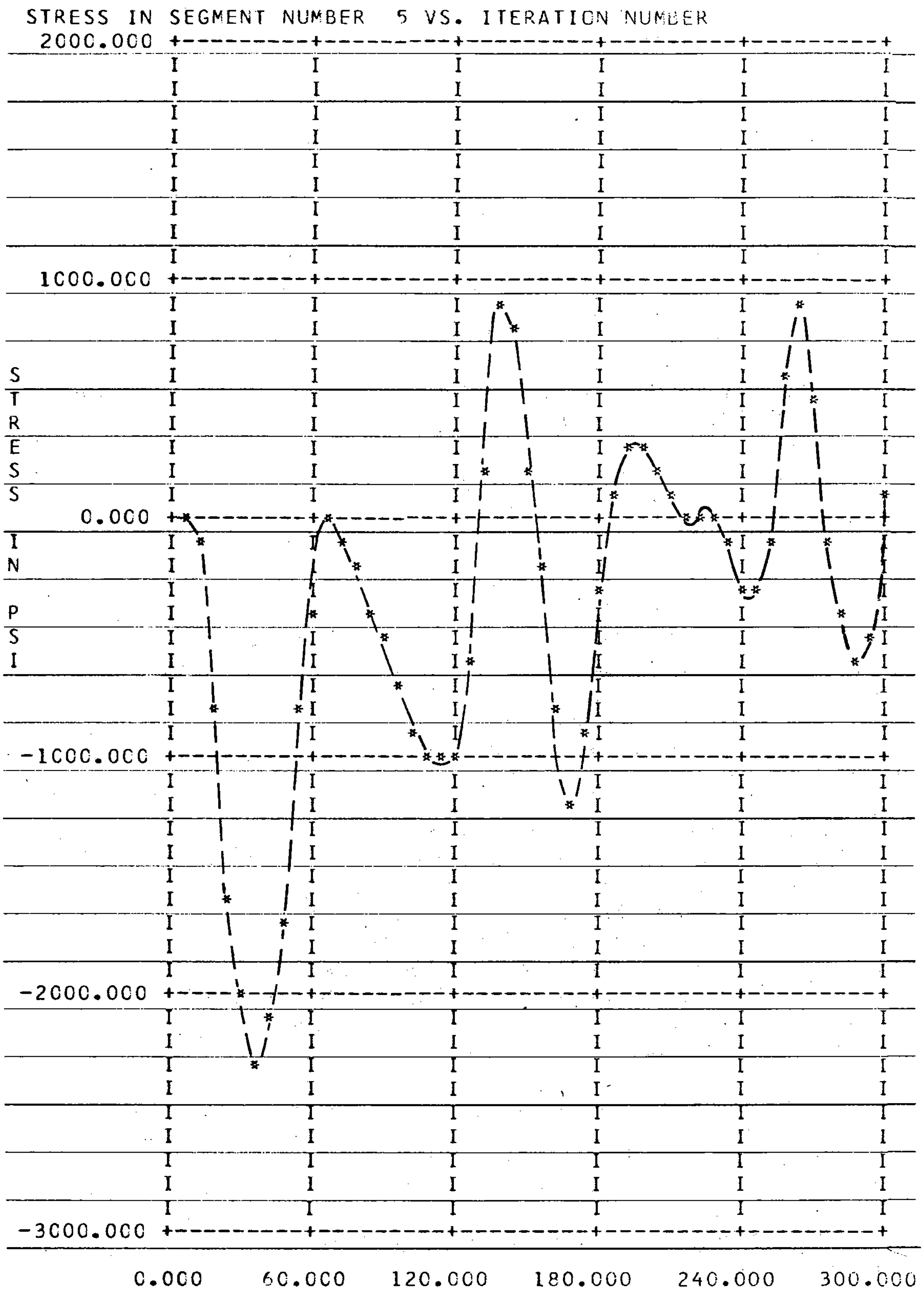

FIG. 33.—COMPUTER-PLOTTED VALUES OF STRESS VERSUS TIME INTERVAL

hysteresis loop. Of course, the value can be adjusted to approximate a known material behavior. By using Eq. 34 and a value of B = 0.0016 in. sec per ft, the theoretical curve of Fig. 30 is obtained. The load-deformation diagram is shown in Fig. 2(c). By comparing the theoretical solutions with test results, a reasonably good correlation is seen over some ranges. In light of the limited information available concerning the dynamic behavior of soil, the writers consider the correlations to be encouraging. In order to compare directly the three theoretical solutions, Fig. 31 is provided.

A series of hysteresis loops taken from a problem involving Eq. 34 is shown in Fig. 32. At the present stage of development, Eq. 34 appears to offer the most realistic means of simulating the internal damping characteristics. The load-deformation relationship of Fig. 2(d) has been suggested by Smith[22] as another possible means of accounting for internal damping. However, no solutions have been obtained using this type of diagram.

Automatic Plotting by Computer.—With solutions of the type obtained by the wave-theory program, graphical representations of results are often desirable. Fig. 33 is provided in order to illustrate the automatic plotting capability of the digital computer. The particular plot given corresponds to the theoretical solution of Fig. 28. A dashed line has been sketched through the asterisk points provided by the computer.

CONCLUSIONS

Although the investigation was not intended to provide a comprehensive parameter study, it is possible to make certain observations with respect to the data presented, as long as the scope of such observations is restricted to the ranges of variables considered.

On the basis of the present investigation, the writers have formed the following conclusions:

1. In the studies presented, good numerical accuracy was obtained by dividing piles into at least ten segments with no segment length exceeding 10 ft and keeping the time interval less than one-half the critical value. For relatively short piles, the number of segments can likely be decreased. It should also be noted that the maximum time interval may be controlled by conditions such as a steel ram striking a stiff capblock.
2. For the cases considered, the gravity effect does not appear severe as far as compressive stresses and permanent sets are concerned. Its influence on tensile stresses can prove significant, however, for prestressed concrete piles.
3. For the limited number of cases in which the gravity effect was considered, there were no significant differences obtained by Smith's approximate method or the influence-coefficient method. However, a much wider range of conditions needs to be studied before any general conclusion can be reached.
4. The influence of explosive pressure on results does not appear pronounced.

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[22] Personal correspondence dated December 15, 1962, from E. A. L. Smith to Charles H. Samson, Jr.

5. The velocity of the ram is of primary importance in the magnitudes of stresses developed.

6. The modulus of elasticity of the pile material is significant with respect to stresses produced.

7. Wave theory as applied to stress propagation offers a rational approach to the problems associated with the structural behavior of piling. It appears that all the significant factors may be provided for.

8. The increasing availability of high-speed digital computers to the engineering profession should make the wave-theory approach more attractive. Future improvements and developments in computer technology will add to the advantages of the procedure.

9. Through generalized parameter studies much useful information may be generated for use by the practicing engineer. It should be possible to develop approximate "rules-of-thumb" for use in determining favorable driving conditions. For example, with a given prestressed concrete pile and a given type of soil, it is of interest to know what hammer characteristics would be most favorable with respect to driving while avoiding excessive tensile stresses.

10. Future work is especially needed to understand the dynamic behavior of soils and the internal damping characteristics of pile materials. This should contribute greatly to obtaining closer agreement of theoretical and test results.

11. Research is needed to study the possibility of relating soil data and the resistance to penetration obtained by wave theory to the ultimate bearing capacity of the pile. As stated previously, it is not the intent of this paper to consider this problem. However, it would seem logical that ultimate bearing capacity would be in some way related to resistance to penetration and other soil properties.

12. Of particular interest are the high tensile stresses found for many of the cases studied. For prestressed concrete piles the implications are obvious, and the matter of the amount of tensile stress developed takes on great significance in the design of the pile and in the specification of driving conditions. Certain practical aspects of this problem have been examined by G. C. Strobel and John Heald,[23] Charles H. Samson, Jr.,[24] and Donovan H. Lee.[25]

13. It is of some interest to compare the rate of hammer blows with the time required for a tensile wave to be reflected back for a long pile. As noted by Samson,[24] a rate of 105 blows per minute allows a time interval of 0.57 sec between blows. Both the theoretical and test results of the present investigation (see Fig. 28) show that this time is well under 0.02 sec, or 1/28 the time between blows. This would seem to show that, at least for piles of the length considered, successive blows cannot be relied on to reduce tensile stresses.

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23 "Theoretical and Practical Discussion of the Design, Testing and Use of Pretensioned Prestressed Concrete Piling," by G. C. Strobel and John Heald, Journal, Prestressed Concrete Inst., Vol. 6, No. 3, September, 1961, p. 22.

24 Discussion by Charles H. Samson, Jr. of "Theoretical and Practical Discussion of the Design, Testing and Use of Prestressed Concrete Piling," G. C. Strobel and John Heald, Journal, Prestressed Concrete Inst., Vol. 6, No. 3, September, 1961, p. 22.

25 Discussion by Donovan H. Lee of "Theoretical and Practical Discussion of the Design, Testing and Use of Pretensioned Prestressed Concrete Piling," by G. C. Strobel and John Heald, Journal, Prestressed Concrete Inst., Vol. 6, No. 3, September, 1961, p. 22.

ACKNOWLEDGMENTS

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This work has served as the basis for a current project being sponsored jointly by the Bridge Division of the Texas Highway Department and the Bureau of Public Roads.

APPENDIX.—NOTATION

The following symbols have been adopted for use in this paper:

$B(m)$ = damping constant for internal spring m, in inches;

$C(m,t)$ = compression of internal spring m in time interval t, in inches;

$C(m,t)_{max}$ = temporary maximum value of C(m,t), in inches;

c = velocity of propagation of stress wave;

$D(m,t)$ = displacement of element m in time interval t, in inches;

$D'(m,t)$ = plastic displacement of external spring m in time interval t, in inches;

d_{mn} = displacement influence coefficient;

E = modulus of elasticity;

$e(m)$ = coefficient of restitution of internal spring m;

$F(m,t)$ = force in element m in time interval t, in pounds;

g = acceleration due to gravity, in feet per second squared;

$J(m)$ = damping constant for soil at element m, in seconds per foot;

$K(m)$ = spring constant associated with internal spring m, in pounds per inch;

$K'(m)$ = spring constant of external spring m, in pounds per inch;

m	=	element number;
p	=	number of m at point of pile;
$Q(m)$	=	quake of external spring m, in inches;
$R(m,t)$	=	force exerted by external spring m on element m in time interval t in pounds;
$Ru(m)$	=	ultimate ground resistance for external spring m, in pounds;
t	=	time, or time interval;
u	=	displacement of a bar cross section in x direction;
$V(m,t)$	=	velocity of element m in time interval t, in feet per second;
$W(m)$	=	weight of element m, in pounds;
x	=	direction of longitudinal axis of a bar;
ρ	=	mass per unit volume;
ΔL	=	length of segment;
Δt	=	size of time interval, in seconds;
$(\Delta t)_{cr}$	=	size of critical time interval, in seconds;
$( )$	=	functional designation;
$[ ]$	=	rectangular matrix;
$\{ \}$	=	column matrix; and
$[ ]^{-1}$	=	inverse of a matrix.

KEY WORDS: computers; concrete, prestressed; foundations; piles; structural engineering

ABSTRACT: The application of wave theory to the investigation of structural behavior of piling is examined. A digital-computer program based substantially on the work of Smith was used in generating the theoretical solution. The essentials of Smith's development are given, followed by an investigation of certain extensions and applications of the procedure. It is illustrated how, through the use of high-speed digital computation, the influence on pile behavior of factors such as ram weight, ram velocity, diesel-hammer pressure, capblock and cushion-block stiffnesses, pile material properties, and soil properties may be evaluated. Comparisons are made with the "exact" solution for an ideal bar and with experimental results from a field test. The effects of segment length and time interval for the discrete-element solution are examined. The use of the automatic plotting capability of the digital computer is illustrated.

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